# John Horton Conway: The Man and His Knot Theory

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ABSTRACT

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John Horton Conway was a British mathematician in the twentieth century. He made notable achievements in fields such as algebra, number theory, and knot theory. He was a renowned professor at Cambridge University and later Princeton. His contributions to algebra include his discovery of the Conway group, a group in twenty-four dimensions, and the Conway Constellation. He contributed to number theory with his development of the surreal numbers. His Game of Life earned him long-lasting fame. He contributed to knot theory with his developments of the Conway polynomial, Conway sphere, and Conway notation.

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I want to thank my advisor Dr. Frederick Norwood for introducing me to the history of math. Historical context brings life to art, and it has been an invaluably rewarding experience to learn about the lives of the mathematicians who gave us a better understanding of our world.

I would like to thank my committee members Dr. Robert Gardner and Dr. Anant Godbole for inspiring me to teach mathematics in an approachable but rigorous way. I would also like to acknowledge a former professor, Dr. Robert Beeler, who encouraged my pursuit of mathematics by responding to the age-old question of “Why do we learn math?” with “What is mathematics but a way for us to make sense of the chaos that perpetuates throughout the universe.”

Lastly I want to thank my parents and my older brother for all of their support throughout the years. My father instilled in me the importance of taking pride in your work, and I wish he were here today to see my accomplishments. My mother, who is steadfast against adversity, inspires me to keep moving forward, and my brother, who I can confide in during the best and worst of times, keeps me grounded and reminds me of the goodness in life.

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# 1 BIOGRAPHY

## 1.1 Early Life and Journey to Cambridge

John Horton Conway was an English mathematician known for his contributions to numerous fields including knot theory and abstract algebra. Born in Liverpool, England on December 26, 1937, Conway's early life was met with challenges characteristic of wartime. Conway's mother, Agnes Conway, worked managing the household. His father, Cyril Horton Conway, worked many jobs over the years, eventually managing the chemistry lab at the Holt High School for Boys. He served as an air raid warden when World War II began, and developed a makeshift telephone system to keep the family's Liverpool neighborhood connected when air raid sirens prompted residents to move to bomb shelters. Conway eventually had to be evacuated alone to Bangor, Wales to wait out the worst of the bombings. After the war ended in 1945, Conway was able to return to his home in Liverpool.

Conway attended the Holt High School for Boys, a non-boarding secondary institute in Liverpool [25]. While there, he was known for his early fascination with British mathematician Coxeter's work *Regular Polytopes* [11]. He pursued independent studies on the Platonic solids, which were published in an article in the school's regular publication, the *Holt School Magazine*, titled "$n$-Dimensional Regular Polytopes" [2]. Conway would continue research in the Platonic solids later in his career with his publication *Sphere Packings, Lattices, and Groups* [10]. Conway also studied knot theory and had enumerated around four thousand knots having eleven or fewer crossings. He later served as secretary for the Science Society, an extracurricular club dedicated to discussing scientific discovery.

In late September of 1956, Conway left his Liverpool home on a train bound for Cambridge. As the only student from his graduating class at Holt High School attending the university, it was this moment that would serve as an inflection point in his life. While in his younger years Conway was known to be a quiet reserved child, he decided on the train to Cambridge that he would shed that persona and try his hand at being extraverted and free-spoken.[1] This personality shift would characterize Conway for the rest of his career, endearing him to his colleagues and students.

## 1.2 Cambridge

As an undergraduate student at Cambridge, Conway was known for pursuing studies outside of the traditional curriculum. In his first term, Conway contributed in his own way to the newly emerged computer age by inventing a water-based computer, known as WINNIE, or Water Initiated Numerical Number Integrating Engine, that could calculate a sum of up to one hundred twenty-seven. Conway was a member of two Cambridge clubs: the Archimedeans and the New Pythagoreans. The New Pythagoreans was associated with five other colleges in the area, while the Archimedeans was a Cambridge specific organization. Conway received his B.A. in 1959, which he followed into graduate studies with a research scholarship.

In late 1962, he was elected a college fellow, giving him a position on the college council. Approaching the end of his graduate studies, Conway began work on his dissertation, working with advisor Harold Davenport. His first attempt at a disserta-

[1] Roberts, Siobhan, Genuis At Play: The Curious Mind of John Horton Conway, Bloomsbury USA, 2015, 22-24

tion began in number theory with work on the problem of expressing numbers as the sum of fifth powers [25]. Conway never submitted his work on this problem, instead finishing his Ph.D. with a work titled "Homogeneous Ordered Sets" [6].

### 1.3 Early Career

Shortly after finishing his Ph.D., he was given a position at Cambridge as an assistant lecturer in the newly formed Department of Pure Mathematics. In addition, Sidney Sussex College also offered him a position. He maintained both positions for the next six years before leaving his fellowship at Sidney Sussex College in 1970 for a full-time professorship at Cambridge. Conway's first major mathematical discovery involved the field of group theory. He discovered three groups in twenty-four dimensions, later to be named the Conway groups. Together the groups form the Conway Constellation [25]. A *group* is an algebraic structure involving a set $G$. Set $G$ is an unordered collection of elements such that each element only appears once. Set elements are not necessarily numbers. Elements of the Conway group involve vectors, i.e. mathematical objects that have both magnitude and direction. In a group, set $G$ is closed under a binary operation, denoted $*$. The binary operation in groups is defined as either addition or multiplication. In set theoretic language, a binary operation $*$ is defined on a set $S$ as a function mapping $S \times S$ into $S$. Consider elements $a, b \in S$. Then for each element $(a, b) \in S \times S$ we define element $*((a, b))$ of $S$ as $a * b$. Closure in a set defined under a binary operation is defined such that for all $a, b \in S$, $a * b \in S$. The following properties define set $G$ as a group. For all $a, b, c \in G$, $(a*b)*c = a*(b*c)$, $\exists e \in G$ such that for all $x \in G$, $e*x = x*e = x$, and for each $a \in G$, there is an element $a' \in G$ such that $a * a' = a' * a = e$. The number of elements in a group is called its order. The largest of the Conway groups has order 8,315,553,613,086,720,000. The discovery began Conway's professional publication career and appeared in the first volume of the *Bulletin of the London Mathematical Society* in 1969 [3].

Additionally in 1969, Conway presented his findings on an idea called the surreal numbers. He further developed the concept in his 1976 publication *On Numbers and Games* and went on to illustrate surreal numbers as a game in his 1996 publication *The Book of Numbers.* Conway begins his definition of the surreal numbers by an abstract concept of simplest. Conway's Simplicity Rule states that if there is any number that fits, the answer is the simplest number that fits [9]. He defined a surreal number as the simplest number between two numbers. The real numbers are defined as a subset of the surreals. The surreals give an infinite class of numbers [7]. In the late 1800s, German mathematician Georg Cantor introduced the idea of two distinct systems of infinite numbers, *cardinal numbers* and *ordinal numbers.* His developments sought to construct a coherent theory of counting collections that may be infinitely large. Cantor denoted the earliest number greater than all the finite counting numbers as $\omega = 0, 1, 2, \ldots, |$ where the vertical bar represents the place where the number sequence is cut off. For example, $0, 1, 2| = 3$ so then $0, 1, 2, \ldots, \omega| = \omega + 1$. Cantor denoted each ordinal number following $\omega$ as $\omega + k$ where $k = 1, 2, \ldots$ to simplify notation. Cantor discovered that adding infinite numbers sometimes does not satisfy the commutative property. Begin by considering addition of finite numbers. For example, $2 + 3 = 3 + 2 = 5$. In *The Book of Numbers*, Conway illustrates how addition of infinite numbers can lead to conflicting results. Using Cantor's notation, $2 = 0, 1|$. Conway represented this visually by a series of parallel vertical lines. In the case of a finite number like two, each vertical line is the same height. For the number two then we have two vertical lines side by side, the left line representing zero and the right line representing one. Then since $3 = 0, 1, 2|$, to add three, we place three more lines starting from the right of the original two. For number three then, one line represents zero, the next represents one, and the last line represents two. An example of this illustration is given in Figure 1 (left) below. $3 + 2$ then is represented by Figure 1 (right) following the same process. For addition of finite numbers, each line will always be equidistant to lines left and right, in both directions.

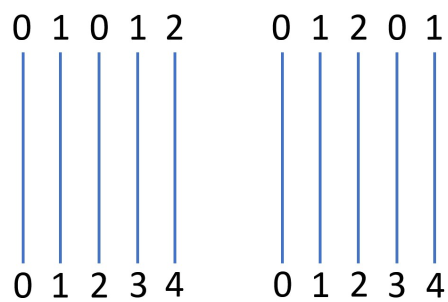


Figure 1: An example demonstrating Conway's method of illustrating the addition of finite collections of numbers using Cantor's notation.

The numbers on top of each vertical line represent the original counting following the process above. The numbers on bottom represent the new counting method using Cantor's notation. We can see in Figure 1 (left) that by the new counting, $2 + 3 = 0, 1, 2, 3, 4| = 5$ and Figure 1 (right), $3 + 2 = 0, 1, 2, 3, 4| = 5$. Thus we conclude that $2 + 3 = 3 + 2$. For addition of infinite numbers, Conway illustrated the process in a similar manner except that for each subsequent number, the height of the line becomes shorter and the distance between lines lesser. The pattern creates the shape of a right triangle. This concept is illustrated below in Figure 2.

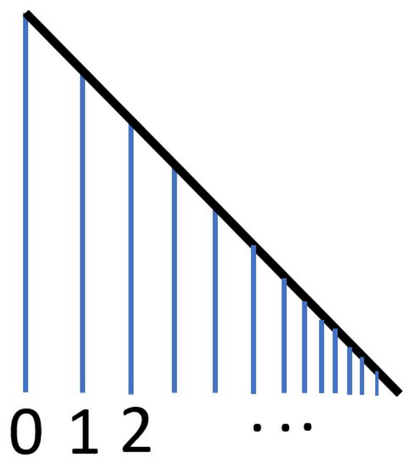


Figure 2: Conway's method of visualizing an infinite collection of numbers.

Consider addition using ordinals such as $1 + \omega$. By following Conway's visual method, given below in Figure 3, we see in the new counting that $1+\omega = \omega$. This is a contradictory result as we know that $\omega+1 > \omega$. So we have that $1+\omega \neq \omega+1$. [8]

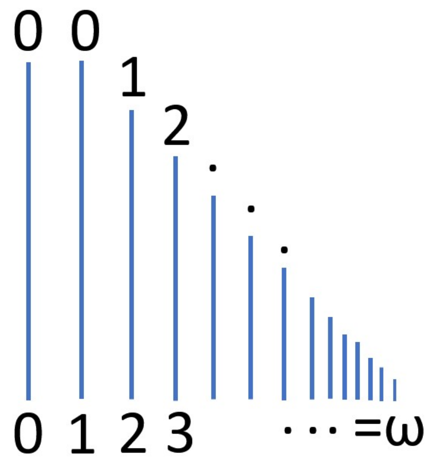


Figure 3: Conway's method to determine $1 + \omega$.

Cantor posited that there were an infinite number of ordinal numbers, a result that was proved decades later by one of Cantor's students, Ernst Zermelo, who developed a principle called the *axiom of choice*. [30] The axiom of choice, according to Conway, says that "if you have any collection of nonempty sets of things, you can make a new set by choosing just one from each set of the given collections." [8] To counter the fact that using the ordinal numbers to count the same set in different ways can often give contradictory results, Cantor developed the cardinal numbers. In the cardinal numbers, counting a set in different ways always gives the same result. This is because, by definition, two collections $A$ and $B$ have the same cardinal number only when there exists a one-to-one correspondence, or a bijection, between them [24]. A *bijection* is a function mapping that is both an injection and a surjection. An *injection* is defined as a function mapping $A$ to $B$, such that for $a_1, a_2 \in A$, where $a_1 \neq a_2$, $f(a_1) \neq f(a_2)$. A *surjection* is a function mapping $A$ to $B$ such that for all $b \in B$, $\exists\, a \in A$ such that $f(a) = b$ [19]. Using Cantor's notation, Conway defined a surreal number by starting with a collection of numbers $A = a, b, c, d, e, f, \ldots$ where $a < b < c < d < e < f < \cdots$ . Then $g \in a, b, c, \ldots, |d, e, f, \ldots$ is the simplest number strictly greater than all numbers in the collection $B = a, b, c, \ldots$ and strictly less than all the numbers in the collection $C = d, e, f, \ldots$ . Conway further illustrated the concept of simplest by a game that appears in *The Book of Numbers*. Suppose a game is played between two players, Left and Right. The game begins by players selecting any number $g$. The number selected is agreed upon by both players as the starting point for the game. The game takes place as a series of single moves. One player begins from $g$ to any number such that it belongs to a particular collection. The rules require that moves from any number to an element $b \in B$ are legal only for Left, and moves from $g$ to an element $c \in C$ are legal only for Right. Suppose both players have agreed upon an initial $g$ and that Left will make the first move. Left moves from $g$ to $b_i$. Left's turn is now over and Right will make a move beginning at $b_i$ where Left ended their move. Right can move to any element $c_j$. After Right moves to $c_j$ their move is finished. The game continues until neither player can make a legal move. Whichever player completes the last legal move is the winner, and the number that is the final move is the simplest number strictly greater than all of the numbers in $B$ and strictly less than all of the numbers in $C$. Note that each number moved to can also be defined as the simplest number between two collections [8]. The surreals differ from the real numbers in that between any two real numbers, there is always another real number, their average. The surreal numbers are as of now a pure mathematical idea and have yet to be found in any applied mathematics case, though Conway's Princeton colleague Peter Sarnack and others suggest they will find applications in the future [25].

Shortly after, in 1970, Conway published an article on knot theory titled "An Enumeration of Knots and Links, and some of their Algebraic Properties" [4]. One of his numerous findings became known as the Conway notation. Conway notation became a powerful tool in determining the construction of knots using building blocks called tangles and applying a numbering system to their construction. Conway identified tangles by developing the Conway sphere. He also discovered an alternative to the standard Alexander polynomial for determining the inequality of knots. This knot invariant became known as the Conway polynomial. Knots, tangles, and other knot theory definitions are given in Sections Two and Three. Knot theory is part of the mathematical area called topology. This article marked Conway's first publication in knot theory [25].

Perhaps Conway's most famous development is his version of a single player game called The Game of Life. The game involves an infinite grid and pieces of two colors. The squares within the grid are called cells and the pieces, usually represented as circles, will inhabit the cells throughout the game. A cellular automaton is a model of computation that evolves in a discrete way. The ideas behind Conway's Game of Life originated from earlier work in simulating cellular automata by mathematicians such as John von Neumann and Stanislaw Ulam. The two distinctly colored pieces are used to represent a cell as being either live or dead. Each cell will either be live or dead, indicated by the single colored piece contained within the cell. Every cell in the grid will contain a piece. A single live cell or a collection of two or more live cells and their live neighbors (i.e. cells sharing an edge or vertex) are called an organism.

The game takes place over a series of discrete moves. The initial configuration of live and dead cells represents time $t = 0$. The base game has three rules: the birth rule, the death rule, and the survival rule. The birth rule says that if at a time $t$ a cell is dead and that cell has three live cell neighbors in any direction, then at time $t + 1$ that cell becomes live. A live cell will become dead per the death rule if it has only one or no live cell neighbors, or if it has four or more live cell neighbors. If at time $t$ a live cell has two or three live cell neighbors, then that cell will stay live at time $t + 1$ per the survival rule. Conway's version of The Game of Life follows the three basic rules plus an additional three rules. The fourth rule is that there should be no initial pattern for which a simple proof can devise a way for the cell population to grow without limit. The fifth rule states that there should be initial patterns that appear to grow without limit, and the sixth rule states that there should be a simple initial pattern that grows and changes for a considerable period of time before coming to an end in one of three possible ways. Those ways are that the organism fades away completely, settles into a stable configuration, or enters into an oscillating phase in which the live cells repeat an endless cycle of two or more time periods. The game does not have any official end point, and so provides an open ended experience with unlimited possibilities for cellular development. [15].

Conway received a full professorship in 1989 after publishing the ATLAS, a detailed collection of finite simple groups, which he coauthored with several other mathematicians [5]. The ATLAS required a decade of work and was the result of the largest collaborative effort in the mathematics community as of its publication.[2]

[2]Roberts, Siobhan, Genuis At Play: The Curious Mind of John Horton Conway, Bloomsbury USA, 2015, 261

## 1.4 Princeton and Later Career

Shortly after the publication of the ATLAS, Conway accepted an invitation to give a talk at Princeton. After his concluding remarks, the chair of the university's mathematics department, Elias Stein, asked Conway about his future at Cambridge. He offered him a position at Princeton, a full time position. Conway accepted a visiting professorship for the 1986-87 academic year which would turn into a full professorship. After three decades at Cambridge, Conway would spend his remaining years at Princeton.

Conway struggled initially to find his place at Princeton. His first year was given to teaching foundational mathematics courses to undergraduates and later to graduate students. In his second year, a colleague of Conway's, William Thurston, approached Conway about his work in group theory. Thurston had been conducting research on the crystallographic groups. He had developed a topological approach to the subject and on telling Conway his ideas they became fast friends. Conway and Thurston, along with their colleague Peter Doyle, began work on strengthening the pedagogy within Princeton's math department [25]. They began co-teaching a course that utilized some of the pedagalogical techniques mathematician David Hilbert had developed in his work "Geometry and the Imagination," namely giving students a non-constrained approach to problem solving [18]. Later on Thurston became one of the leading topologists of the twentieth century, thanks to his work on hyperbolic structures on 3-manifolds [29].

In April of 1992, Conway was elected a Fellow of the American Academy of Arts and Sciences for his work in mathematics and education. Conway retired from Princeton in the early 2010s. He still gave the occasional visiting lecture, but had largely settled into a life of reflection. He had spent five decades studying mathematics and left behind a number of contributions. He died in 2020 from complications of COVID-19 at the age of 82 [25].

# 2 KNOT THEORY PREREQUISITES

## 2.1 Topology

Knot theory lies in the mathematical field of topology [21]. To understand the mathematical properties of knots, one must first understand some basic concepts in topology. A fundamental problem in topology involves determining whether two spaces are homeomorphic [13] [23]. To define a homeomorphism, we must define some topological spaces. The set of all ordered triples of real numbers is called 3-space, or $\mathbb{R}^3$ [28]. The $n$-sphere, or $n$-dimensional sphere, is the unit sphere in $\mathbb{R}^{n+1}$ [17]. A *circle* is a 1-sphere in $\mathbb{R}^2$ and the shape commonly called a *sphere* is a 2-sphere in $\mathbb{R}^3$. An *open ball* of radius $r$ and center $p$ is the set of all points whose distance from $p$ is strictly less than $r$. It is that strictly less than that makes the ball open. If we said less than or equal to the ball would be *closed*. A set is called open if every point in the set is contained in an open ball inside the set. We can determine that two spaces are *homeomorphic* if there exists a continuous function with a continuous inverse mapping one space onto the other. A function $f$ is said to be *continuous* if the inverse image of an open set is open. Let $X$ and $Y$ be two topological spaces. Then let $f : X \to Y$ be a bijection. If both the function $f$ and the inverse function $f^{-1} : Y \to X$ are continuous, then $f$ is called a *homeomorphism* [23]. In knot theory, knots exist in ambient space. *Ambient space* is the space surrounding a mathematical object along with the object itself. The ambient space in knot theory in three dimensions is given by $\mathbb{R}^3$ or $S^3$. When considering $\mathbb{R}^3$, if we add a point at infinity, then a process called *one point compactification* will give $S^3$, that is, $S^3 = \mathbb{R}^3 \cup \{\infty\}$. Because knots are closed and bounded, it does not matter which of these two spaces, $\mathbb{R}^3$ or $S^3$, we use for the ambient space. The difference between them is that $S^3$ has an additional point denoted as $\infty$. Knots need not pass through this additional point. Therefore, if two knots are equivalent in $S^3$ then they are also equivalent in $\mathbb{R}^3$, and vice versa.

Consider two distinct points $p$ and $q$ in three-dimensional space. We denote a line segment joining the two points as $[p, q]$. Now consider an ordered set of $n$ distinct points, given by $(p_1, p_2, \ldots, p_n)$. The union of the line segments $[p_1, p_2], [p_2, p_3], \ldots, [p_{n-1}, p_n]$ and $[p_n, p_1]$ is called a *polygonal (or piecewise linear) curve*. A piecewise linear curve is said to be closed if there exists a line segment from $p_n$ to $p_1$. If each line segment intersects exactly two other segments only at their end points, then the curve is called a *simple curve*. Simple closed curves will be used to define knots. [22]

## 2.2 Knot Theory Definitions

A *link* $L$ of $m$ components is a subset of $S^3$ (or of $\mathbb{R}^3$) that consists of $m$ disjoint, piecewise-linear simple closed curves. A link of one component is a *knot*.

We say $L$ is in three-dimensional space, but when considering an image of $L$ we determine a projection of $L$ into two-dimensional space. Links are often considered as a set of simple closed curves, though links can also be determined using polygons with many short sides that are arbitrarily close to the curve. Consider a triple $(x, y, z)$ in three-dimensional space and a pair $(x, y)$ in two-dimensional space. A function that maps $(x, y, z)$ to $(x, y)$ is called the *projection map*. The image of $L$ under the projection map is called the *projection of $L$*. A link projection is called a *regular projection* if no three points on the link project to the same point, and no vertex projects to the same point as any other point on the link. Regular projections guarantee that no information about a link has been lost, provided we indicate at each double point of the projection which segment is higher and which is lower. The higher segment is called an *overcrossing*, the lower an *undercrossing*. An example of a knot projection that is not regular is given in Figure 4 (right) below. Figure 4 demonstrates the usefulness of regular projections in link diagrams. If the left knot were rotated slightly, we would lose information about the projection. In this example, we would lose information on one of the simple closed curves.

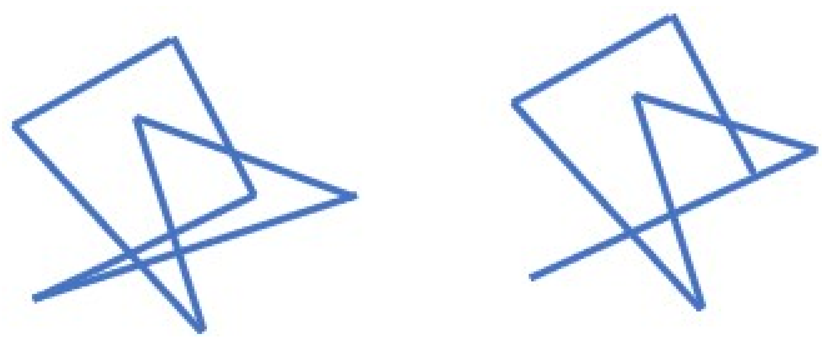

Figure 4: An example of a regular and non-regular knot projection.

A link that is projected onto 2-space (i.e. two-dimensional space), loses information on its overcrossings and undercrossings. Then to illustrate crossings, over and undercrossings are represented by breaks in the curves. Note in Figure 5 (right) below, a diagram of the trefoil knot, that some curves have breaks while others do not. A curve that has a break before a crossing indicates that the curve crosses under the other curve while the other curve crosses over. We note the alternating over and undercrossings of the strands. A knot is called the *unknot* if it is equivalent to a knot whose diagram has no crossings. A diagram of the unknot is given in Figure 5 (left).

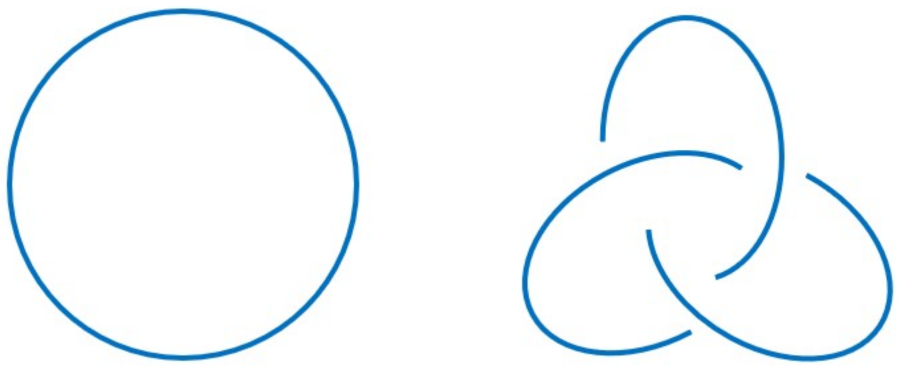

Figure 5: An example of knot diagrams of the unknot & trefoil knot.

Consider a link $T$. $T$ is said to be tame if it has a diagram with finitely many crossings. Consider the case that $T$ does not have a diagram with finitely many crossings. If every diagram of $T$ has infinitely many crossings it is called wild.

*Reidemeister moves* allow for manipulations of a knot diagram that result in equivalent knots. The three types of Reidemeister moves are given in Figure 6 below [21].

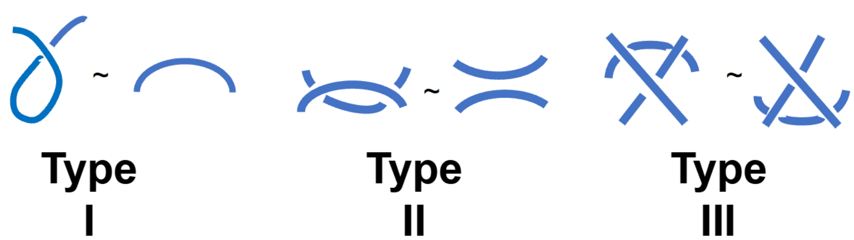


Figure 6: A diagram that illustrates the Reidemeister moves.

In 1926, Kurt Reidemeister proved that given two distinct projections of the same knot, by utilizing the Reidemeister moves and planar isotopies, we can without loss of generality arrive at one projection from the other. A *planar isotopy* of a knot projection is a continuous deformation of the plane onto which it is projected. The result is achieved by using a homeomorphism of the plane, which is a continuous mapping of the plane onto itself which sends a regular diagram to another regular diagram. An example is given in Figure 7 below [1].


Figure 7: A diagram of the Reidemeister moves and distinct projections from Adams. (1994, p. 14)

An *invariant* is a well-defined mathematical entity such as a number, a polynomial, or a group. Invariants are useful in knot theory to demonstrate that two knots are not homeomorphic. Reidemeister moves can be used to determine certain knot invariants. [1].

In 1969, Conway devised a link invariant called the *Conway polynomial* (related to another polynomial called the *Alexander polynomial* of a link) which can be calculated using a skein relation. A *skein relation* gives an equation which relates the polynomial of a link to the polynomial of links obtained by changing the crossings of the projection of the original link. Simple closed curves are given by a union of line segments such that each segment intersects exactly two other segments only at their endpoints. This set of intersections gives a set of vertices of the link. We apply a direction on the set of vertices of each component of a link by putting an arrow on it.

A skein relation is an equation that utilizes the orientation of curves as they cross to define positive, negative, and zero crossings. Consider a diagram of an oriented link $L$. If, in a small neighborhood of a crossing, the orientation of two curves meeting at a crossing shows one curve passing over the other curve from left to right, then that crossing is positive, denoted $L_+$. If the orientation shows one curve passing over the other from right to left, that crossing is negative, denoted $L_-$. If the diagram near a crossing is redrawn, so the two curves keep their orientation but no longer crossing, the resulting diagram is denoted $L_0$. A diagram for a skein relation is given in Figure 8 below.

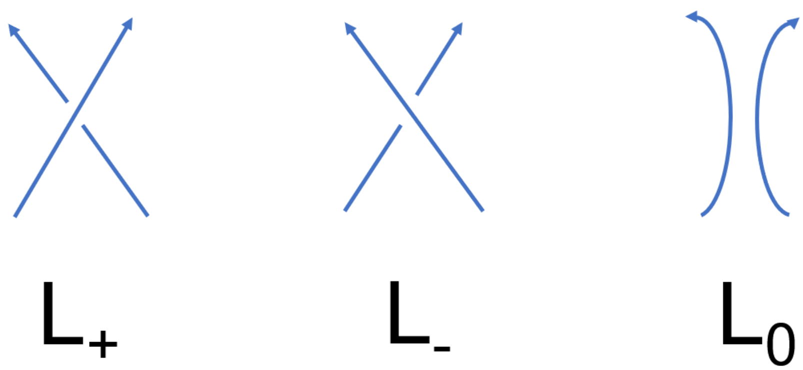


Figure 8: A diagram of a skein relation.

Note that in changing from $L_+$ to $L_0$, the direction of the arcs is preserved. Consider a skein relation of the trefoil knot, denoted as $K$ and given in Figure 9 below.

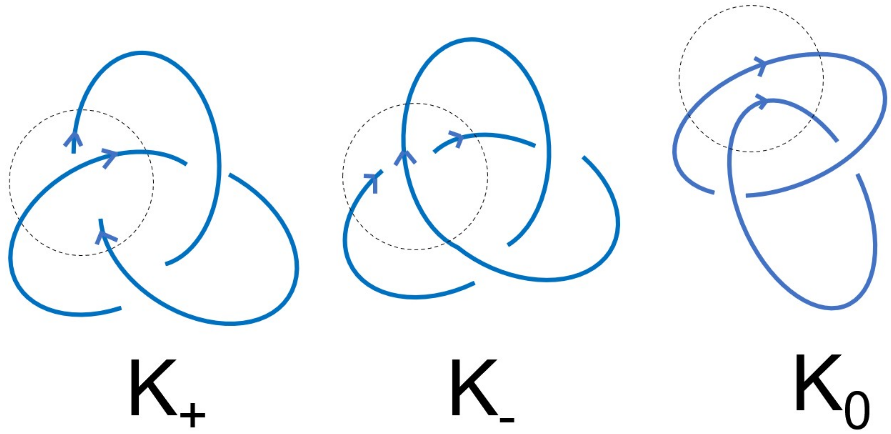


Figure 9: A skein relation of the trefoil knot.

By utilizing Reidemeister moves, we note that $K_-$ unravels to the unknot. $K_0$ is called the Hopf link.

# 3 CONWAY'S CONTRIBUTIONS

## 3.1 Conway Polynomial

The Conway polynomial, a polynomial with integer coefficients, is a link invariant that provides an alternative to the Alexander polynomial. Although two links can have the same Conway polynomial, it does not necessarily indicate that they are homeomorphic. However if two links have different Conway polynomials, they cannot be homeomorphic. The Conway polynomial can be useful in determining *chiralities* for knots. Consider an oriented knot $K$. By reversing the orientation of $K$, we get another oriented knot called its *reverse*, denoted $K^r$. If $K$ is equivalent to $K^r$ using Reidemeister moves, it is called reversible. Changing all of the crossings of $K$ yields the *mirror image* of $K$, denoted $K^m$. If $K$ is equivalent to $K^m$ using Reidemeister moves, it is called *amphicheiral*. If we were to change all of the crossings of $K$ and reverse its orientation, we would obtain $K^{rm} \equiv K^{mr}$. If $K$ is equivalent to $K^{rm}$, it is called *negative amphicheiral*.

The Conway Polynomial of the unknot equals one. The Conway polynomial for an oriented knot $K$, denoted $\nabla_K(z)$, is calculated using the skein relation formula $\nabla_{K_+}(z) - \nabla_{K_-}(z) = z\nabla_{K_0}(z)$, or equivalently, $\nabla_{K_+}(z) = z\nabla_{K_0}(z) + \nabla_{K_-}(z)$. For an oriented link, replace $K$ by $L$. For the unknot, denoted $U$, we have that $\nabla_U(z) = 1$. A link is called *splitable* if, after suitable Reidemeister moves, there is a plane in $\mathbb{R}^3$ so that not all of the components of the link are on the same side of the plane. The Conway polynomial of an oriented link $L$ has the property that if $L$ is a split link, then $\nabla_L(z) = 0$.

We will calculate the Conway polynomial for the trefoil knot as follows: Let $K$ denote the trefoil knot. It was previously noted that by utilizing Reidemeister moves, $K_-$ unravels to the unknot. Then by Conway's skein relation formula, we have $\nabla_{K_+}(z) = z\nabla_{K_0}(z) + 1$. We must next determine the Conway polynomial for the Hopf link, which we will denote as $H$. Note, $K_0 \equiv H$, which gives that by Conway's skein relation formula, $\nabla_{K_0}(z) \equiv \nabla_H(z)$. We give a skein relation of the Hopf link in Figure 10 below.

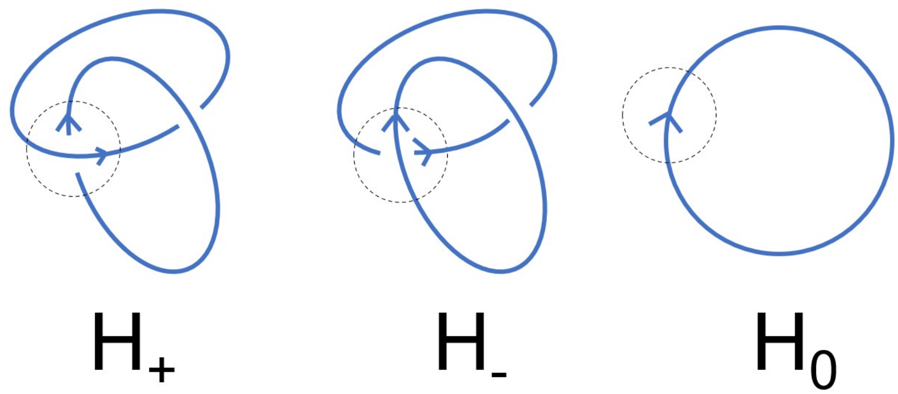


Figure 10: A skein relation of the Hopf link.

Now we have that $\nabla_{H_+}(z) = z\nabla_{H_0}(z) + \nabla_{H_-}(z) = z$. This is because $H_0 \equiv U$ and thus $\nabla_{H_0}(z) \equiv \nabla_U(z) = 1$. And $H_+$ is a split link, giving that $\nabla_{H_+}(z) = 0$. By this then the Conway polynomial for the trefoil knot is given by $\nabla_{K_+}(z) = z\nabla_{K_0}(z) + 1 = z\nabla_{H_0}(z) + \nabla_{H_-}(z) + 1 = z^2 + 1$.

### 3.2 Alexander Polynomial

A Laurent polynomial differs from an ordinary polynomial in that where ordinary polynomials cannot have negative powers of the variable, a Laurent polynomial may have negative powers. For an oriented link $L$, the Alexander polynomial of $L$, denoted $\Delta_L(t)$, is a Laurent polynomial associated with the link in an invariant way. The Alexander polynomial of $L$ is a family of equivalent Laurent polynomials, as opposed to Conway's polynomial of $L$ which is an ordinary polynomial [21]. The coefficients of both Conway polynomials and Alexander polynomials are integers. The *units* (i.e. elements having a multiplicative inverse) of a Laurent polynomial are the powers of the variable [12]. The Alexander polynomial can be found using the equation $\Delta_{L_+}(t) - \Delta_{L_-}(t) = (1-t)\Delta_{L_0}(t)$. Since the Alexander polynomial gives a family of equivalent Laurent polynomials, an equivalent Alexander polynomial for $L$ can be found by multiplication by any unit.

An Alexander polynomial of a link can be calculated from its Conway polynomial by substituting $z = t^{1/2} - t^{-1/2}$. Laurent polynomials are not defined to have fractional exponents, so after substitution we must follow the laws of exponents to reach whole number powers. [22].

## 3.3 Conway Sphere

The *Conway sphere* is an essential element in determining how knots are constructed. To develop the concept, we revisit crossings and projections. A link diagram for a link $L$ is a projection of $L$ onto a plane. In regular projections, every crossing is a double crossing. Below in Figure 11 is a projection of a double crossing (left) and a triple crossing (right). The figure below depicts transverse crossings. A transverse crossing in $L$ is a crossing where one strand passes across another strand, that is, where they intersect, they cross, rather than just touching and then turning back.

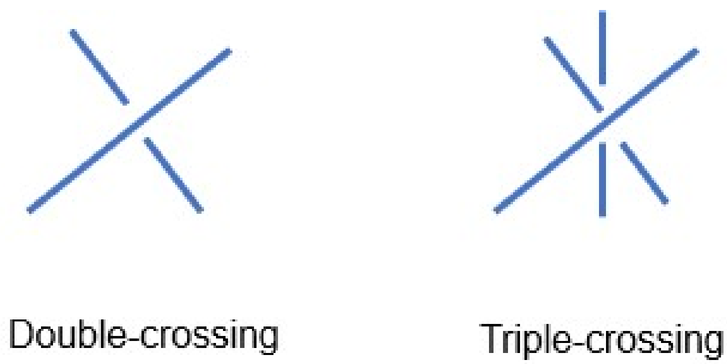


Figure 11: An example of a double and triple crossing projection.

To add to the definition of a split link, we note that a link $L \subset S^3$, having at least two components, is a split link if there is a 2-sphere in $S^3 - L$ separating $S^3$ into two balls, each containing a component of $L$. An example of a split link is given in Figure 12 below.

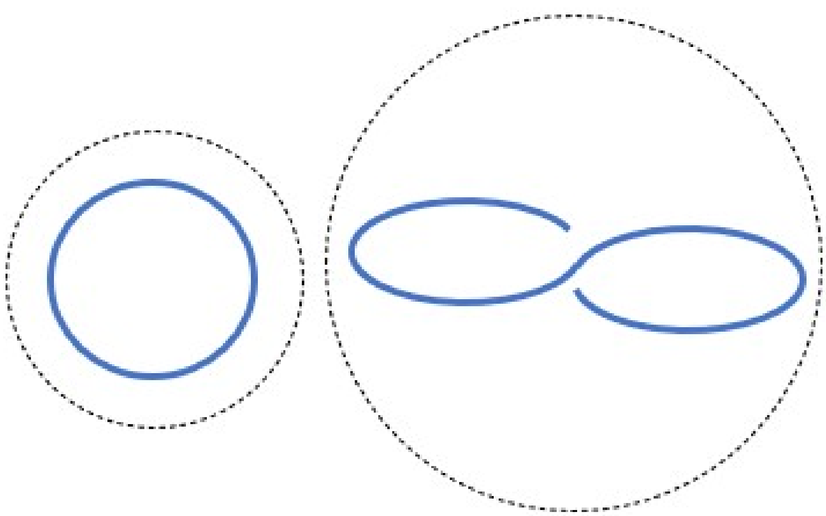

Figure 12: An example of a split link.

To understand a Conway sphere, we need the definition of an *incompressible surface* $F$ in a 3-manifold $M$ and also the definition of a *spanning disk* for the surface $F$ in $M$. A spanning disk for $F$ in $M$ is a disk in $M$ that intersects $F$ only in the boundary of the disk, which is a circle. We can picture a spanning disk as a blister above $F$ whose boundary is on $F$. "We now define an incompressible surface $F$ in $M$. There are two cases to consider. If $F$ is a 2-sphere, it is incompressible if it does not bound a ball in $M$. In the case where $M$ is a link complement, this will happen only when the link is separable, with one part inside $F$ and the other part outside $F$. The more important case occurs when $F$ is not a 2-sphere. It helps to picture $F$ as a sphere with several points removed, leaving holes in $F$ through which a link $L$ bounds. In this case $F$ is incompressible when the boundary of any spanning disk also bounds a disk that lies entirely on the surface $F$. This happens when every spanning disk either covers none of the holes in $F$ or covers all of the holes in $F$. A Conway sphere for a link $L$ in $S^3$ is a 2-sphere $\Sigma$ in $S^3$ that meets $L$ transversely at four points such that (i) $\Sigma - L$ is incompressible in $S^3 - L$ and (ii) any 2-sphere in $S^3 - \Sigma$ meeting $L$ transversely at two points bounds a ball in $S^3 - \Sigma$ meeting $L$ in just an unknotted arc [21]. An example of a Conway sphere is given in Figure 13 below.

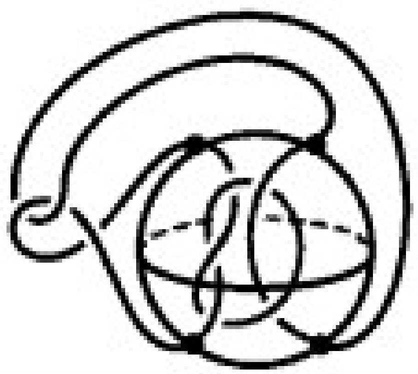

Figure 13: An example of a Conway sphere from Adams. (1994, p. 94)

### 3.4 Conway Notation

Conway developed a notation for links that he used to tabulate the prime knots up to eleven crossings and prime links up to ten crossings. A link $L \subset S^3$, excluding the unknot, is prime if every 2-sphere in $S^3$ that intersects $L$ transversely at two points and bounds, on one side of it, a ball that intersects $L$ in precisely one unknotted arc. A composite link is a link that is not prime. An example of a prime and composite knot is given in Figure 14 below [1].

Figure 14: An example of a prime and composite knot.

Given a link diagram for a link $L$, a *tangle* is a region within a Conway sphere.

Two tangles are said to be equivalent if one can utilize Reidemeister moves to arrive at one tangle from another.

A tangle consisting of two strands is called the 0-tangle. The 0-tangle is given in Figure 15 below. We can create a different tangle by utilizing the 0-tangle. We twist the two strands of the 0-tangle around each other three times. This new tangle is denoted as the 3-tangle and is shown in Figure 15 (right).

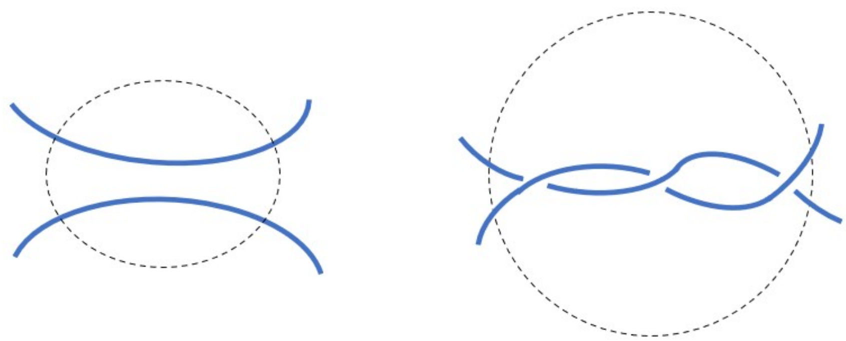

Figure 15: An example of a diagram of the 0-tangle and 3-tangle.

The resulting twists that form the 3-tangle from the 0-tangle are associated with a positive integer three. The twists are denoted as positive by the positive slope of the overcrossings. If we were to form the 3-tangle the other way around, the resulting 3-tangle would be associated with a negative integer three, given by the negative slope of the overcrossings. An illustration of the positive and negative slopes is given in Figure 16 below.

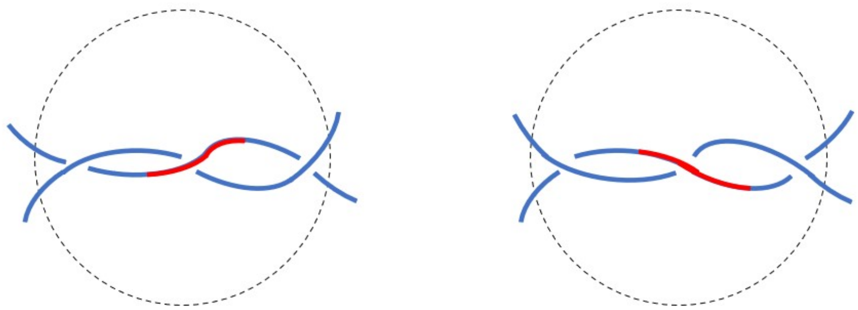

Figure 16: An example of the 3-tangle and $-3$-tangle.

Any tangle that can be constructed with twists associated with integer values is called a *rational tangle.* Consider the 3-tangle given in Figure 17. If we were to reflect the tangle about a northwest to southeast diagonal, we would have the tangle given in Figure 17 (left) below. If we then extend the upper right strand and the lower right strand and give them two twists, we construct a 3 2 tangle given in Figure 17 (middle). If the resulting tangle is again reflected as before, and the new upper right and lower right strands are extended and given a -2 twist, we get a 3 2 -2 tangle given in Figure 17 (right).

Figure 17: An example of the 3-tangle, 3 2 tangle, and 3 2 -2 tangle.

After introducing the Conway notation for rational tangles, Conway gave a simple way to tell when two rational tangles are equivalent [1].

### 3.5 Conclusions

John H. Conway brought notable advances to the field of knot theory. His developments of the Conway skein relation formulas to determine the Conway polynomial of an oriented link gave a useful alternative to the Alexander polynomial. Once one determines the Conway polynomial of an oriented link, a simple substitution of $z = t^{-1/2} - t^{1/2}$ can be used to find the Alexander polynomial of the oriented link. The Conway sphere is useful in determining bounded surfaces in knot theory and for developing the concept of tangles. Conway notation provided ways to tabulate knots based on the number of crossings and ways to construct knots using tangles, providing insights into determining equivalent knots. Methods to determine knot equivalence are a prominent research area of knot theory. Conway's developments provided insights into the symmetries of regular projections of knots. Symmetry does not provide a definitive way to determine equivalence, but his contributions have proven useful in showing some equivalences.

Conway did not dedicate himself to one field of mathematics. He chose to be a part of the greater mathematical community and worked alongside several mathematicians who would become well renowned in their respective fields. In his early life he struggled with social acceptance. He persevered through his undergraduate studies despite competing interests outside of the curriculum and became a prominent social figure in his academic circle. After failing to complete his dissertation on the first attempt, he succeeded in his second and landed a position at Cambridge shortly after.

He was prolific in publication in numerous fields including algebra, number theory, and knot theory. He moved to Princeton in his later career and was an essential element in the development of their mathematics curriculum. He left behind a legacy of academic achievement and discovery.

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