**Collatz conjecture**

The unresolved Collatz conjecture from 1937 states that when the function
\[ f(x) = \begin{cases} x/2 & \text{if } x \equiv 0 \pmod{2} \\ 3x + 1 & \text{if } x \equiv 1 \pmod{2} \end{cases} \]
is iterated on an initial positive integer \( x \) we eventually reach the cycle \((1, 4, 2)\).

- Starting from an initial value \( x \), the sequence of iterates: \( f(x), f(f(x)), \ldots \)
  behaves at first irregularly before its eventual inevitable decline from some power of 2 down to the three element cycle.
- For example, starting the iteration at 12 provides the sequence:
  \[ 12, 6, 3, 10, 5, 16, 8, 4, 2, 1, 4, 2, 1, \ldots \]
- The sequence of numbers up until the first power of two is the interesting phase of the iteration, which we will call its trace.

**Producing permutations**

- The elements of a trace from an initial value \( x \) are all distinct, and we produce a Collatz permutation, \( C(x) \), by replacing the \( i^{\text{th}} \) smallest number of the trace by \( i \).
- So \( C(12) = 5.3.1.4.2 \).
- Clearly, the map \( C \) is not one to one. For instance:
  \[ 1 = C(3) = C(21) = C(85) = \ldots = C((2^3 - 1)/3) = \ldots \]
- However, as the length of \( C(x) \) increases, coincidences become more rare, e.g. the only other \( x \leq 1000 \) with \( C(x) = C(12) \) is \( x = 988 \).

**Enumerating Collatz permutations**

Among the permutations of length \( n \), how many are Collatz permutations?
Considering only those \( x \leq 10^4 \) for which the length of \( C(x) \) is at most 7 produces the following table.

<table>
<thead>
<tr>
<th>Length</th>
<th>Collatz permutations</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
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<tr>
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<tr>
<td>6</td>
<td>8</td>
</tr>
<tr>
<td>7</td>
<td>13</td>
</tr>
</tbody>
</table>

As the reader will have noticed the values in the table are the Fibonacci numbers.

**Types of traces**

- Two kinds of steps occur in a trace depending on the parity of the argument:
  - \( u \) up steps \( (x \mapsto 3x + 1) \) when \( x \) is odd
  - \( d \) down steps \( (x \mapsto x/2) \) when \( x \) is even
- We call the resulting sequence of \( u \)'s and \( d \) the type of the trace. A type of a trace necessarily satisfies:
  - Two up steps can never occur consecutively since \( 3x + 1 \) is even when \( x \) is odd.
  - The last symbol in such a sequence must be a \( d \), since there is a "hidden" \( u \) occurring next to take us to a power of 2. Thus the number of traces of a given length are given by the Fibonacci numbers.
- In order to show that there are at least Fibonacci-many Collatz permutations of length \( n \) it will be enough to show that any sequence of \( u \)'s and \( d \)'s satisfying the necessary conditions above actually occurs as the type of some trace.

**Witnesses for a type**

- The final element of a trace is a number of the form \((A - 1)/3\) where \( A = 2^n \).
  Define the inverses of \( u \) and \( d \):
  \[ U(u) = u^{-1}(x) = (x - 1)/3 \quad \text{and} \quad D(d) = d^{-1}(x) = 2x. \]
- A witness for a type \( \sigma \), is an \( A = 2^n \) such that there is a trace ending at \((A - 1)/3\) with type \( \sigma \). For example, to find a witness for the type \( \text{ududududd} \) requires at least that \( UDDDU(A) \) is an integer. Note here that the "hidden" \( u \) has become explicit.
- Now \( UDDDU(A) = 8A - 28 \)

In order for this to be an integer requires \( A = 7 \) (mod 27), and remembering that \( A = 2^n \), the least solution to this is \( A = 2^{16} = 65536 \).
- Unraveling the applications of \( U \)'s and \( D \)'s leads to \( x = 19417 \). The trace of \( x \) is
  \[ 19417, 58252, 29262, 14564, 43000, 21845, \text{and } C(x) = 26.41.5.3. \]

**At least Fibonacci**

It turns out that the initial necessary integrality condition is also sufficient and that infinitely many witnesses always exist.

**Proposition 1**

If a type \( \sigma \) contains \( u \)'s then there is a single congruence of the form \( A = c \mod{2^{\sigma_{t}}(3^{\sigma_{u}})} \) which must be satisfied in order that a trace of type \( \sigma \) ends with witness \( A \). Consequently, there is a least witness \( A \) with \( a \leq 2 \cdot 3^t \), and a general witness is of the form \( 2^{t+1} \cdot 3^i \) where \( i \) is a nonnegative integer and \( d = 2 \cdot 3^t \).
This proposition shows that every potential type has a witness and thereby proves that there are at least Fibonacci-many Collatz permutations of length \( n \).

**Excess**

There are exactly Fibonacci-many Collatz permutations of length \( n \) for \( n \leq 14 \) but for greater \( n \) there are more.

<table>
<thead>
<tr>
<th>length</th>
<th>#perms</th>
<th>length</th>
<th>#perms</th>
<th>excess</th>
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<tbody>
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<tr>
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<td>28</td>
<td>13</td>
</tr>
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</table>

**The first ET**

- The first type that is associated with more than one Collatz permutation (an excess type or ET) is \( \sigma = ududududdudddd \) which has the integrality condition \( 2^n = 16 \mod 729 \).
- The smallest solution is \( a = 4 \) corresponding to the trace
  \[ 9, 28, 14, 7, 22, 11, 34, 17, 52, 26, 13, 40, 20, 10, 5 \]
- However, the next solution to the integrality condition is \( a = 99 \), giving an initial number with 140 digits producing the second permutation below.

**Explaining the ET’s**

- A potential witness \( i \) for the type \( \sigma = udududduddd \) generates the trace
  \[ UDDDUU(i), UDDDU(i), UDDU(d(i), \ldots, D(i), U(i)). \]
  We can think of these as linear functions in \( i \). We find a witness for the type where a vertical line \( t = A = 2^t \) has intersection points with these lines at integer heights.
- We can then read off the relative order of the lines at this point according to the order they are crossed as we move up the line \( t = A \) from the \( t \)-axis.

**At most 2×Fibonacci**

- Recall that the type \( \sigma = ududududduddudd \) had witnesses \( 2^t \cdot 2^t \). The greatest abscissa of an intersection point for the lines arising from this sequence is approximately 41.04. This explains why we get two different Collatz permutations.
- It turns out that we can get at most two permutations from each type:

**Proposition 2**

For any type \( \sigma \) there are at most two distinct permutations \( C(x) \) arising from \( x \) of type \( \sigma \).

**Open problems and conjectures**

- How exactly does the number \( c_n \) of ET’s of length \( n \) behave? The data suggests that it might be something like "half Fibonacci rate" i.e. \( c_n \approx c_{n-1} + c_{n-2} \).
- We always get \( i = 16 \) in the integrality conditions for ET’s. Is this just bias in the data that we currently have?
- We have an extrinsic way of creating the Collatz permutations: run the Collatz process and see what comes out. Is there an intrinsic way to recognize these permutations?