The "Seesaw Method":
An Easy Way to Work Mixture Problems
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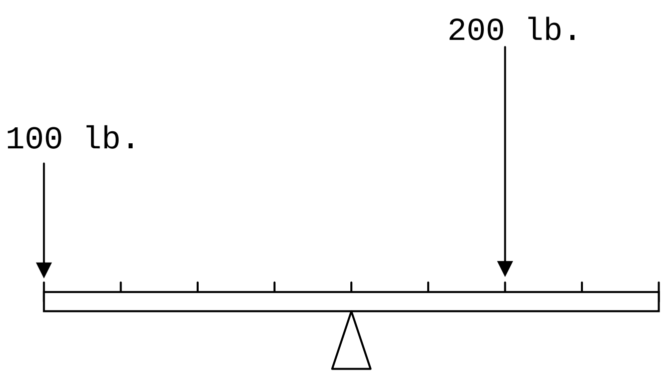
Some word problems require us to mix two quantities of different concentrations to produce a mixture whose concentration is to be some specific value between the other two. To solve such problems more easily, I developed a "Seesaw Method" from a principle of Engineering Statics. My students find this much easier to use than traditional methods. First I will introduce the idea of the Seesaw Method using an illustration; then I will show examples of how to apply this to mixture problems; finally, I will offer a proof of this method.

To illustrate the principle of the Seesaw Method, suppose we have an 8-foot seesaw (teeter-totter) with the fulcrum (point of balance) in the middle. A 100-lb. boy sits at one end, 4 feet from the fulcrum. Where should a 200-lb. man sit on the other side to balance him out? Answer: Two feet from the fulcrum! Why? Because their *torques* must be equal in order for the seesaw to be balanced. (The torque is the person's weight multiplied by his distance from the fulcrum.) See the diagram below:



Boy's torque = Man's torque

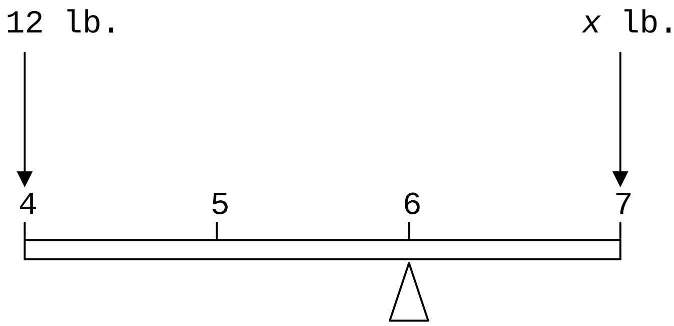
(100 lb.)(4 ft.) = (200 lb.)(2 ft.)

400 ft-lb. = 400 ft-lb.

That is, each person is exerting 400 foot-pounds of torque about the fulcrum. But these two torques are acting in opposite directions (the man's torque is clockwise and the boy's torque is counterclockwise); so they cancel each other out and the seesaw is balanced. Now let's apply this analogy to an example:

How many pounds of coffee worth \$7 per pound must be mixed with 12 pounds of coffee worth \$4 per pound to make a mixture worth \$6 per pound?

Let's let x = the number of pounds of the \$7/lb. coffee that is needed. We make a "Concentration Seesaw" that shows the dollar-per-pound "concentration" of the coffees. We put the low concentration (\$4/lb.) at the left end, and the high concentration (\$7/lb.) at the right end. (See the diagram below.) We have 12 pounds of the \$4/lb. coffee and x pounds of the \$7/lb. coffee; so we have 12 lb. of coffee "acting" at 4, and x lb. of coffee "acting" at 7. Now, the resulting mixture must be worth \$6/lb.; that is, we want the dollar-per-pound "concentration" of the mixture to "balance out" at \$6/lb. -- so we put the fulcrum at 6. The 12 lb. weight is acting 2 units from the fulcrum (since 4 and 6 are 2 units apart on the seesaw), and the x lb. weight is similarly acting 1 unit from the fulcrum. Since the two "torques" must be equal, we have:



$$(12 \text{ lb.})(2) = (x \text{ lb.})(1)$$

$$24 = x$$

(Notice that it was not necessary to write the "\$/lb." units for the numbers on the seesaw: since they are the same on both sides of the equation, they will cancel out anyway.)

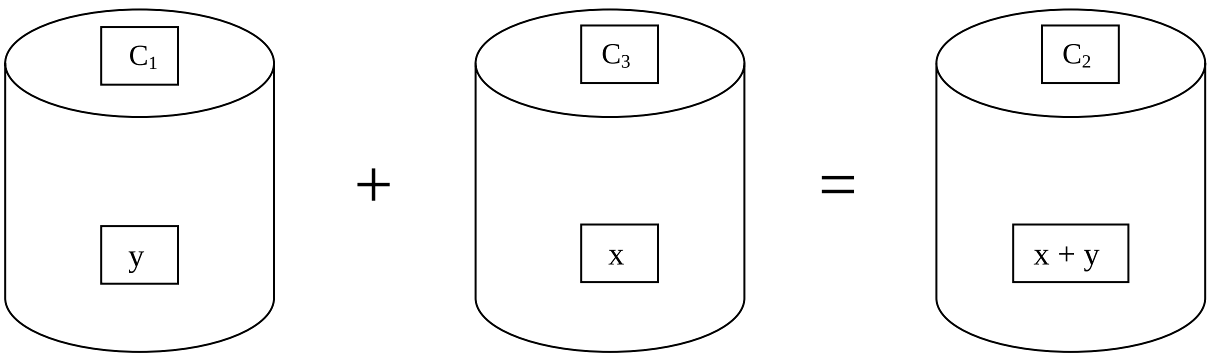
As a similar example, suppose we are asked, "**How many gallons of a 70% alcohol solution must be mixed with 12 gallons of a 40% alcohol solution to make a mixture that is 60% alcohol?**" Using the figure above, we would change "lb." to "gal."; also, "4" through "7" on the seesaw would be changed to "40%" through "70%". So our equation would now read

$$(12 \text{ gal.})(20) = (x \text{ gal.})(10)$$

Hence, $x = 24$

(Note: If "pure water" is mentioned in the problem statement, this would equate to a 0% solution, and "pure alcohol" would equate to a 100% solution.)

A Proof of the Seesaw Method. Suppose we are asked, "How many gallons of a C_3% alcohol solution must be mixed with *y* gallons of a C_1% alcohol solution to make a mixture that is C_2% alcohol?", where $C_1 < C_2 < C_3$. Or, we may be similarly asked, "How many pounds of coffee worth C_3 dollars per pound must be mixed with *y* pounds of coffee worth C_1 dollars per pound to make a mixture worth C_2 dollars per pound?" In either case, let x represent the unknown quantity we are being asked to find. One traditional method for working these problems is the "Barrel Method", illustrated below. It shows that two barrels of concentrations C_1 and C_3 are being added together to make a mixture of concentration C_2. Note that the concentration is written on the lid of each barrel, and the quantity is written inside of the barrel.



Now two relationships must be preserved across the equals sign. First, the *quantity* of the mixture (the barrel on the right) must be the sum of the quantities in the two barrels on the left; hence, we add the *y* and the *x* to get the "*x + y*" in the barrel on the right. Secondly, the *amount of pure alcohol* in the mixture must be the sum of the amounts of pure alcohol in the other two barrels. [Or, the dollar value of the coffee mixture must be the sum of the dollar values of the coffees in the other two barrels.] This latter relationship gives us the equation

$$C_1 y \ + \ C_3 x \ = \ C_2(x+y)$$

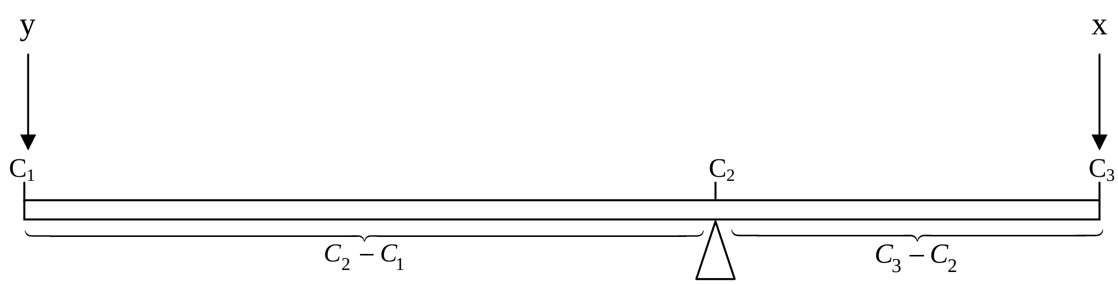
$$C_1 y \ + \ C_3 x \ = \ C_2 x \ + \ C_2 y$$

$$C_3 x \ - \ C_2 x \ = \ C_2 y \ - \ C_1 y$$

$$(C_3 - C_2)x \ = \ (C_2 - C_1)y$$

Hence, $$x = \frac{C_2 - C_1}{C_3 - C_2} \ y \, .$$

Now let's set up this same example using the Seesaw Method. First, we construct a horizontal bar representing a seesaw (teeter-totter). The units on the seesaw are concentration-related. The low concentration, C_1, is placed at the far left, and the high concentration, C_3, is placed at the far right. The fulcrum (point of balance) is placed at C_2, since we want the concentration of the resulting mixture to "balance out" at C_2. Now *y* -- the quantity of concentration C_1 -- is written above the C_1 on the seesaw, and *x* -- the quantity of concentration C_3 -- is written above the C_3 on the seesaw. See the diagram below.



Now a principle of Engineering Statics tells us that "the sum of all external torques acting on a rigid body in equilibrium must equal zero". Hence, in order for the seesaw to be balanced, the "torques" of the two quantities *x* and *y* must be equal and opposite. (Each "torque" is the quantity multiplied by its distance from the fulcrum.) Now *y*'s torque is acting (counterclockwise) a distance of $C_2 - C_1$ units from the fulcrum, and *x*'s torque is acting (clockwise) a distance of $C_3 - C_2$ units from the fulcrum. Since the magnitudes of the two torques must be equal, we have

$$(C_2 - C_1)\, y \;=\; (C_3 - C_2)\, x$$

Hence, $$x = \frac{C_2 - C_1}{C_3 - C_2}\, y,$$

the same result obtained by the Barrel Method.

To explain the principle of the Seesaw Method, let's return for a moment to our original illustration of the boy and the man on the seesaw. If they wanted to sit the same distance away from the fulcrum, they would have to weigh the same. But suppose the man wants to sit closer to the fulcrum. The shorter his distance from the fulcrum, the more he must weigh so that his torque -- the product of his weight and distance from the fulcrum -- will be the same as the boy's. The Seesaw Method is entirely analogous -- except that C_1 and C_3 are fixed positions, and it is C_2 (the fulcrum's position) which moves. If we wanted the concentration of the mixture to be exactly half way between the concentrations of the two quantities we are mixing, we would obviously need equal amounts of those two quantities (that is, *x* and *y* would be equal). But the *greater* we want the concentration of the mixture to be (that is, the closer C_2 moves to C_3), the more we need of the quantity of *higher* concentration, C_3 (that is, the greater *x* must be).

Finally, it should also be mentioned that a similar proof and explanation would ensue if x represented the (unknown) quantity of *lesser* concentration (C_1) and y represented the (known) quantity of greater concentration (C_3). In that case, either method (Barrel or Seesaw) would give us the result

$$x = \frac{C_3 - C_2}{C_2 - C_1} y .$$