1.5 Column Space and Nullspace

Definition: the *coefficient matrix* for a system of equations is the augmented matrix with its last column deleted. The *column space* of a matrix is the span of the columns (treated as vectors) of the matrix.

Note: If the columns of the coefficient matrix are $A_1,\ A_2,\ \ldots,\ A_n$ then a linear combination of these (i.e. an element of the column space) is of the form

$$B = x_1A_1 + x_2A_2 + \cdots + x_nA_n$$

Theorem 1 *A linear system is solvable iff the vector of constants belongs to the column space of the coefficient matrix.*

Definition: A nonempty set W of some vector space is a *subspace* if it is "closed under linear combinations." This means that for every $X,\ Y\ \in W$, all linear combinations of X and Y also belong to W.

Notation: We represent the system of equations

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & + & \vdots & + & \ddots & + & \vdots & = & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m. \end{array}$$

as $AX = B$ where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Definition: If A is a matrix, then the set of vectors X such that $AX = 0$ is the *nullspace* of A

Theorem 2 *The nullspace of an $m \times n$ matrix A is a subspace of $\mathbb{R}^n$.*

Proof:

(1) $\overrightarrow{0} \in$ nullspace $\implies$ nonempty

(2) The nullspace is closed under linear combinations.