

### 3 Linear Transformations

#### 3.1 Linear Transformations

**Note:** Let  $X = [xy]^T$  be a vector from the origin to the point  $(x,y)$  in  $\mathbb{R}^2$ . Then with

$$R_\psi = \begin{bmatrix} \cos \psi & -\sin \psi \\ \sin \psi & \cos \psi \end{bmatrix}$$

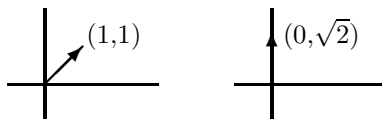
the vector  $R_\psi X$  is obtained from  $X$  by rotating counterclockwise about the origin through an angle  $\psi$ .

**Example:** Consider  $X = [1 \ 1]^T$  and  $R_{\pi/4} = \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}$ .

Then

$$\begin{aligned} R_{\pi/4}[1 \ 1]^T &= \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} \sqrt{2}/2 - \sqrt{2}/2 \\ \sqrt{2}/2 + \sqrt{2}/2 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ \sqrt{2} \end{bmatrix} \end{aligned}$$

Graphically,



**Definition:** A transformation is a rule for mapping elements of one set (the domain of the transformation) into another set (the target space).

**Definition:** A transformation  $\mathbf{T}$  with domain  $\mathbb{R}^n$  and target space  $\mathbb{R}^m$  which can be represented by matrix multiplication is called a matrix transformation (notice that such a matrix would be  $m \times n$ .)

**Proposition 1** Matrix transformations transform line segments onto line segments.

**Proof:** Consider the line segment from  $X$  to  $Y$ . Parametrically this line is

$$tX + (1-t)Y = Y + t(X - Y)$$

where  $t \in [0, 1]$ . Let  $A$  be the matrix of the transformation. Then the image of this line segment is

$$\begin{aligned} A(Y + t(X - T)) &= AY + At(X - Y) \\ &= AY + tA(X - Y) \\ &= AY + tAX - tAY \\ &= t(AX) + (1 - t)(AY). \end{aligned}$$

So the image is a line segment from  $AX$  to  $AY$ .

QED

**Definition:** Let  $\mathcal{V}$  and  $\mathcal{W}$  be vector spaces and let  $T : \mathcal{V} \rightarrow \mathcal{W}$  be a transformation. Then  $T$  is a linear transformation if:

- (a)  $\forall X, Y \in \mathcal{V}, T(X + Y) = T(X) + T(Y)$ , and
- (b)  $\forall X, Y \in \mathcal{V}$  and all scalars  $c, T(cX) = cT(X)$ .

### Matrix Representation Theorem

Let  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  be a linear transformation. Then there is a unique matrix  $A$  such that  $T(X) = AX \quad \forall X \in \mathbb{R}^n$ .

**Proof:** Let  $X = [x_1 x_2 \cdots x_n]^T \in \mathbb{R}^n$  and let  $\{I_1, I_2, \dots, I_n\}$  be the standard basis for  $\mathbb{R}^n$ . Then

$$\begin{aligned} X &= x_1 I_1 + x_2 I_2 + \cdots + x_n I_n \\ \text{and } T(X) &= T(x_1 I_1 + x_2 I_2 + \cdots + x_n I_n) \\ &= x_1 T(I_1) + x_2 T(I_2) + \cdots + x_n T(I_n). \end{aligned}$$

If we define matrix  $A$  such that it has columns  $T(I_1), T(I_2), \dots, T(I_n)$  then  $T(X) = AX$ .

To show uniqueness we assume that another matrix  $B$  also represents  $T$  and show that  $A$  and  $B$  are the same matrix (see p. 140). QED