4.2 Projection: The Gram-Schmidt Process

Definition: Let B and Q be vectors, $Q \neq \vec{0}$. The *projection* of B onto Q is

$$proj_Q(B) = B_0 = \frac{B \cdot Q}{Q \cdot Q} Q.$$

Let $Q_1, Q_2, \ldots, Q_m$ be an *orthogonal* basis for a subspace $\mathcal{W}$ of $\mathbb{R}^n$and let $B \in \mathbb{R}^n$. The *projection* of B onto $\mathcal{W}$ is

$$proj_{\mathcal{W}}(B) = B_0 = \frac{B \cdot Q_1}{Q_1 \cdot Q_1} Q_1 + \frac{B \cdot Q_2}{Q_2 \cdot Q_2} Q_2 + \cdots + \frac{B \cdot Q_m}{Q_m \cdot Q_m} Q_m.$$

Fourier Theorem The vector $proj_{\mathcal{W}}(B) = B_0$ is the unique vector in $\mathcal{W}$ such that $B_1 = B - B_0$ is perpendicular to $\mathcal{W}$.

Note: The Gram-Schmidt process allows us to take a basis $A_1, A_2, \ldots, A_n$ for $\mathbb{R}^n$and produce an orthogonal basis for $\mathbb{R}^n$.

Gram-Schmidt Process

Let $A_1, A_2, \ldots, A_n$ be a basis for $\mathbb{R}^n$. We produce an orthogonal basis as follows:

(a) Let $Q_1 = A_1$.

(b) Let $B_2 = proj_{Q_1}(A_2) = \frac{A_2 \cdot Q_1}{Q_1 \cdot Q_1} Q_1$.

(c) Let $Q_2 = A_2 - B_2$.

(d) Let $B_3 = proj_{span\{Q_1, Q_2\}}(A_3) = \frac{A_3 \cdot Q_1}{Q_1 \cdot Q_1} Q_1 + \frac{A_3 \cdot Q_2}{Q_2 \cdot Q_2} Q_2$.

(e) Let $Q_3 = A_3 - B_3$.

(f) etc.

In general, for $k \ = 0$:

$$\begin{aligned} \text{Let } B_{k+1} &= \frac{A_{k+1} \cdot Q_1}{Q_1 \cdot Q_1} Q_1 + \frac{A_{k+2} \cdot Q_2}{Q_2 \cdot Q_2} Q_2 + \cdots + \frac{A_{k+i} \cdot Q_k}{Q_k \cdot Q_k} Q_k \\ &= proj_{span\{A_1, Q_2, \ldots, Q_k\}}(A_{k+1}), \end{aligned}$$

and let $Q_{k+1} = A_{k+1} - B_{k+1}$.

Notice: By the Fourier Theorem, the new set $\{Q_1, Q_2, \ldots, Q - n\}$ is an orthogonal basis for $\mathbb{R}^n$.

Theorem 1 *Every subspace of $\mathbb{R}^n$has an orthogonal basis.*

Theorem 2 *Let $\mathcal{W}$ be a subspace of $\mathbb{R}^n$ and let $\{Q_1, Q_2, \ldots, Q_n\}$ form an ordered orthonormal basis for $\mathcal{W}$. For each $X \in \mathcal{W}$, let X' denote the coordinate vector of X WRT their ordered basis. Then*

(a) The transformation that transforms each element of X of $\mathcal{W}$ to its coordinate vector X' is a one-to-one transformation of $\mathcal{W}$ onto $\mathbb{R}^k$.

(b) For all $X, Y \in \mathcal{W}$ and for all scalars c,

$$(X + Y)' = X' + Y'$$
$$(cX)' = cX'.$$

(c) For all $X, Y \in \mathcal{W}$

$$X \cdot Y = X' \cdot Y'$$
$$|X| = |X'|.$$

Note: Theorem 2 states that sums of vectors, scalar multiplication, dot product, and length of vectors are "the same", regardless of the orthonormal basis used for the vector space.

Definition: Two vector spaces are isomorphic if they are the same, only the vectors are (possibly) labeled differently in the two spaces.

Note: Theorem 2 then implies:

The Fundamental Theorem of Vector Spaces

Any n-dimensional vector space is isomorphic to $\mathbb{R}^n$.