6 Diagonalization and Matrix Representation

6.1 Eigenvectors

Definition: Let A be a $n \times n$ matrix. A non-zero vector X such that $AX = \lambda X$ for some scalar λ is called an *eigenvector* for A. The scalar λ is called the *eigenvalue*.

Note: If λ is an eigenvalue of A and X is a corresponding eigenvector, then

$$\begin{aligned} AX - \lambda X &= 0 \\ (A - \lambda I)X &= 0 \end{aligned}$$

If $A - I\lambda$ is invertible, then $X = 0$, but X is non-zero by definition. So, $A - I\lambda$ must NOT be invertible and $\det(A - I\lambda) = 0$.

Definition: For $n \times n$ matrix A, the (n-degree) polynomial $p(\lambda) = \det(A - I\lambda)$ is the *characteristic polynomial* of A. The roots of $p(\lambda)$ are the eigenvalues of A.

Note: If X_1 andX_2 are eigenvectors corresponding to λ (i.e. $AX_1 = IX_1$ and $AX_2 = IX_2$) then a linear combination of X_1 and X_2 is an eigenvector corresponding to λ:

$$\begin{aligned} A(aX_1 + bX_2) &= A(aX_1) + A(bX_2) \\ &= aAX_1 + bAX_2 \\ &= a\lambda X_1 + b\lambda X_2 \\ &= \lambda(aX_1 + bX_2) \end{aligned}$$

Therefore the eigenvectors corresponding to eigenvalue λ form a vector space called the *eigenspace corresponding to λ*