Osculating Paths and Oscillating Tableaux

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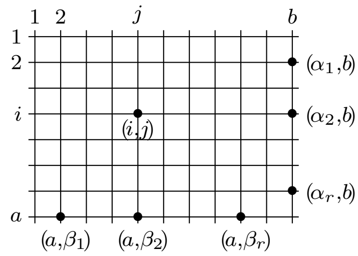
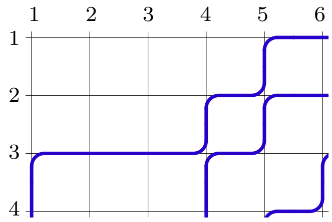
In this talk, the combinatorics of osculating lattice paths will be considered, and it will be shown that there are bijections between certain tuples of osculating paths and certain generalized oscillating tableaux. Full details of this work, and further results, can be found in [1].

The main aims of this work are first to generalize well-known bijections, such as those of Sect. 6 of [2], between nonintersecting paths and semistandard Young tableaux, and second to improve the understanding of the combinatorics of alternating sign matrices, for example to explain combinatorially the appearance of determinants in enumeration formulae derivations such as that of [3], and possibly to find correspondences, such as those conjectured in [4], with certain plane partitions.

The geometric configuration being considered is an a by b rectangle of lattice points with rows labeled 1 to a from top to bottom, columns labeled 1 to b from left to right, and the point in row i and column j labeled (i, j). Furthermore, r points on the lower boundary, $(a, \beta_1), \ldots, (a, \beta_r)$, and r points on the right boundary, $(\alpha_1, b), \ldots, (\alpha_r, b)$, are chosen, as shown below.



Denoting $\{\alpha_1, \ldots, \alpha_r\}$ as α, and $\{\beta_1, \ldots, \beta_r\}$ as β, define $\mathrm{OP}(a, b, \alpha, \beta)$ to be the set of all r-tuples of paths in which the k-th path of a tuple starts at (a, β_k) and ends at (α_k, b), each path of a tuple can take only unit steps upwards or rightwards, and different paths within a tuple are allowed to share lattice points, but not to cross or share lattice edges, i.e., the paths are *osculating*. A running example will be the following element P of $\mathrm{OP}(4, 6, \{1, 2, 3\}, \{1, 4, 5\})$.



There is a straightforward bijection between such path tuples and alternating sign matrices (see for example Sect. 4 of [1]).

Now define the partition

$$\lambda_{a,b,\alpha,\beta} \;:=\; [a]\times[b]\setminus(b-\beta_1,\ldots,b-\beta_r \,|\, a-\alpha_1,\ldots,a-\alpha_r)\,,$$

where Frobenius notation is being used, and for a partition μ with no more than a parts and $\mu_1 \le b$, $[a]\times[b]\setminus\mu$ denotes the complement of μ in the a by b rectangle, $[a]\times[b]\setminus\mu := (b-\mu_a, b-\mu_{a-1},\ldots,b-\mu_1)$. Thus, for example, $\lambda_{4,6,\{1,2,3\},\{1,4,5\}} = [4]\times[6]\setminus(5,2,1\,|\,3,2,1) = [4]\times[6]\setminus(6,4,4,3) = (3,2,2)$. For a path tuple $P \in \mathrm{OP}(a,b,\alpha,\beta)$, define *vacancies* and *osculations* respectively to be points of the a by b rectangle through which no or two paths of P pass. Thus, the running example has vacancies $(1,1)$, $(1,2)$, $(1,3)$, $(1,4)$, $(2,1)$, $(2,2)$, $(2,3)$, $(4,2)$ and $(4,3)$, and osculations $(2,5)$ and $(3,4)$. It is shown in [1] that each $P \in \mathrm{OP}(a,b,\alpha,\beta)$ is uniquely determined by its vacancies and osculations, and that

$$(\text{number of vacancies in } P) - (\text{number of osculations in } P) \;=\; |\lambda_{a,b,\alpha,\beta}|.$$

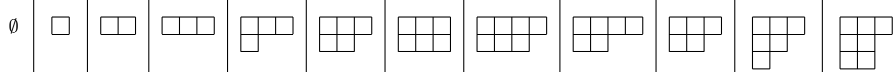
It will be of particular interest here to consider sets of path tuples with a fixed number l of vacancies and osculations,

$$\mathrm{OP}(a,b,\alpha,\beta,l) \;:=\; \Big\{\, P \in \mathrm{OP}(a,b,\alpha,\beta) \;\Big|\; (\text{number of vacancies in } P) \;+\; (\text{number of osculations in } P) \;=\; l \,\Big\}.$$

Now, for a partition λ and nonnegative integer l, an *oscillating tableau* of shape λ and length l is a sequence of $l+1$ partitions starting with $\emptyset$, ending with λ, and in which the Young diagrams of successive partitions differ by a square. Denote the set of all such oscillating tableaux as $\mathrm{OT}(\lambda,l)$. The enumeration of oscillating tableaux is studied in in [5].

For $\eta = (\eta_0,\eta_1,\ldots,\eta_l) \in \mathrm{OT}(\lambda,l)$, define the *profile* of η as $\Omega(\eta) := (j_1-i_1,\ldots,j_l-i_l)$, where (i_k,j_k) is the position of the square by which the Young diagrams of η_k and η_{k-1} differ. It can be checked straightforwardly that each oscillating tableau is uniquely determined by its profile. An example of an element η of $\mathrm{OT}((3,2,2),11)$, with its Young diagrams and profile, is as follows.

k	0	1	2	3	4	5	6	7	8	9	10	11
η_k	$\emptyset$	(1)	(2)	(3)	(3,1)	(3,2)	(3,3)	(4,3)	(4,2)	(3,2)	(3,2,1)	(3,2,2)
$\Omega(\eta)_k$		0	1	2	-1	0	1	3	1	3	-2	-1



Now, for an integer q and positive integer n, define the set $\mathrm{GOT}(n, q, \lambda, l)$ of *generalized oscillating tableaux* to be the set of pairs $((t_1, \ldots, t_l), \eta)$ in which

- t_k is an integer between 1 and n, for each $k = 1, \ldots, l$
- η is an oscillating tableau of shape λ and length l
- $t_k < t_{k+1}$, or $t_k = t_{k+1}$ and $\Omega(\eta)_k \prec_q \Omega(\eta)_{k+1}$, for each $k = 1, \ldots, l-1$, where $\prec_q$ is the total strict order on the integers defined by $z \prec_q z'$ if and only if $|z-q| > |z'-q|$ or $z-q = q-z' < 0$ i.e., $\ldots \prec_q q-2 \prec_q q+2 \prec_q q-1 \prec_q q+1 \prec_q q$.

The main result of this talk is that there is a bijection between $\mathrm{OP}(a, b, \alpha, \beta, l)$ and $\mathrm{GOT}(\min(a, b), b-a, \lambda_{a,b,\alpha,\beta}, l)$, where the generalized oscillating tableau (t, η) which corresponds to a path tuple P is obtained as follows.

(1) For each lattice point (i, j), define $L_{i,j} := \begin{cases} \max(i, j+a-b), & a \le b \\ \max(i-a+b, j), & a \ge b\,. \end{cases}$

(2) Order the l vacancies and osculations of P as $(i_1, j_1), \ldots, (i_l, j_l)$, with $L_{i_k,j_k} < L_{i_{k+1},j_{k+1}}$, or $L_{i_k,j_k} = L_{i_{k+1},j_{k+1}}$ and $j_k - i_k \prec_{b-a} j_{k+1} - i_{k+1}$, for each $k = 1, \ldots, l-1$.

(3) Then $t = (L_{i_1,j_1}, \ldots, L_{i_l,j_l})$ and η is the oscillating tableau with profile $\Omega(\eta) = (j_1 - i_1, \ldots, j_l - i_l)$.

A feature of this bijection is that if (i_k, j_k) is a vacancy of P, then the Young diagram of η_k is related to that of η_{k-1} by the addition of a square, while if (i_k, j_k) is an osculation of P, then η_k is related to η_{k-1} by the deletion of a square. Also, it follows immediately from the bijection that

$$|\mathrm{OP}(a, b, \alpha, \beta, l)| = \sum_{\eta \in \mathrm{OT}(\lambda_{a,b,\alpha,\beta}, l)} \binom{\min(a, b) + A_{b-a}(\eta)}{l},$$

where $A_q(\eta) = |\{k \,|\, \Omega(\eta)_k \prec_q \Omega(\eta)_{k+1})\}|$.

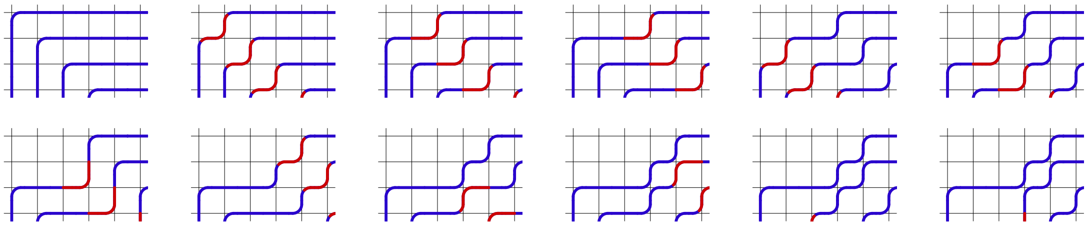
Applying the bijection to the running example P of a path tuple gives the following.

(1) $L_{i,j} = \max(i, j-2)$

(2) The ordered list of vacancies and osculations is $(1, 1)$, $(1, 2)$, $(1, 3)$, $(2, 1)$, $(2, 2)$, $(2, 3)$, $(1, 4)$, $(3, 4)$, $(2, 5)$, $(4, 2)$, $(4, 3)$.

(3) $t = (1, 1, 1, 2, 2, 2, 2, 3, 3, 4, 4)$ and $\Omega(\eta) = (0, 1, 2, -1, 0, 1, 3, 1, 3, -2, -1)$, so η is the previous example of an oscillating tableau, $\eta = \big(\emptyset$, (1), (2), (3), $(3, 1)$, $(3, 2)$, $(3, 3)$, $(4, 3)$, $(4, 2)$, $(3, 2)$, $(3, 2, 1)$, $(3, 2, 2)\big)$.

Some components of the proof of the bijection result will be outlined in the talk. For example, the proof involves introducing, for each path tuple P, a sequence of $l+1$ path tuples in the a by b rectangle which starts with a tuple which has no vacancies or osculations, ends with P, and in which each successive tuple has an additional vacancy or osculation. For the running example P, this sequence is



where the sections of each path tuple which differ from the previous path tuple are shown in red.

Finally, some further, on-going work will be discussed. This may include:

- Using the osculating paths – generalized oscillating tableaux bijection and other known bijections, such as that of [5], to obtain determinantal enumeration formulae and generating functions for alternating sign matrices, and possibly an alternating sign matrix – descending plane partition bijection.
- Studying osculating paths with other external configurations.
- Providing a representation theoretic interpretation of generalized oscillating tableaux.

References

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