Counting on Chebyshev Polynomials (Extended Abstract)

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Chebyshev polynomials arise in a variety of continuous settings. They are a sequence of orthogonal polynomials appearing in approximation theory, numerical integration, and differential equations. In this paper we approach them instead as discrete objects, counting the sum of weighted tilings. Our combinatorial apporach will allow us to prove identities holding for these continuous functions using discrete arguments.

The *Chebyshev polynomials of the first kind* are defined by $T_0(x) = 1$, $T_1(x) = x$, and for $n \geq 2$,

$$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x).$$

The next few polynomials are $T_2(x) = 2x^2 - 1$, $T_3(x) = 4x^3 - 3x$, $T_4(x) = 8x^4 - 8x^2 + 1$, $T_5(x) = 16x^5 - 20x^3 + 5x$.

The *Chebyshev polynomials of the second kind* differ only in the initial conditions. They are defined by $U_0(x) = 1$, $U_1(x) = 2x$, and for $n \geq 2$,

$$U_n(x) = 2xU_{n-1}(x) - U_{n-2}(x).$$

The next few polynomials are $U_2(x) = 4x^2 - 1$, $U_3(x) = 8x^2 - 4x$, $U_4(x) = 16x^4 - 12x^2 + 1$, $U_5(x) = 32x^5 - 32x^3 + 6x$.

Chebyshev polynomials have a simple combinatorial interpretation. We define an n-tiling to be a sequence of squares (of length one) and dominoes (of length two) with a total length on n. For example, there are exactly five 4-tilings, namely $ssss$, ssD, sDs, Dss, and DD, where s denotes a square and D denotes a domino. (It is easy to prove that the number of n-tilings is equal to the n-th Fibonacci number, f_n, where $f_0 = f_1 = 1$, but we won't exploit this fact.) For Chebyshev polynomials of the second kind, we allow our square tiles to come in two colors, light or dark, which we denote by a and b respectively. Each square tile has a *weight* of x and each domino has a weight of -1. We define the weight of a tiling to be the product of the weights of its tiles. For example, the 4-tilings $abba$, baD, and DD have respective weights x^4, $-x^2$, and 1. If we sum the weights of all possible 4-tilings, we have $16x^4 - 12x^2 + 1$, which is $U_4(x)$. Using their recursive definition, one can easily prove by induction the following theorem.

Theorem 1 *For $n \geq 0$, $U_n(x)$ is the sum of the weights of n-tilings using light squares and dark squares, each of weight x, and dominoes of weight -1.*

A similar interpretation exists for Chebyshev polynomials of the first kind, except that we only count tilings that begin with a domino or begin with a light square. (Hence the tiling baD would be bad, since it begins with a dark square.) We call these tilings *restricted*. The tiles have the same weights as before. Observe that there are only eight restricted tilings of the type $ssss$, each with weight x^4, and the total weight of all possible 4-tilings is $8x^4 - 8x^2 + 1 = T_4(x)$.

Theorem 2 *For $n \geq 0$, $T_n(x)$ is the sum of the weights of restricted n-tilings using light squares and dark squares, each of length x, and dominoes of weight -1. The restriction only counts tilings that do not begin with a dark square.*

Using this combinatorial interpretation, many identities involving Chebyshev polynomials become transparent. For example, an n-tiling with exactly k squares will necessarily have $\frac{n-k}{2}$ dominoes, which is an integer if and only if k and n have the same parity. Such a tiling will have $\frac{n+k}{2}$ tiles and there are $\binom{\frac{n+k}{2}}{k}2^k$ ways to create them, each having weight $(-1)^{\frac{n-k}{2}}x^k$, where the binomial coefficient is zero when k and n have opposite parity. As an immediate consequence, we have

Identity 1 *For $n \geq 0$,*

$$U_n(x) = \sum_{k=0}^{n} (-1)^{\frac{n-k}{2}} 2^k \binom{\frac{n+k}{2}}{k} x^k.$$

By examining restricted n-tilings with exactly k dominoes (and considering the initial tile), we get

Identity 2 *For $n \geq 0$,*

$$T_n(x) = \sum_{k=0}^{n} (-1)^{\frac{n-k}{2}} 2^{k-1} \left[\binom{\frac{n+k}{2}}{k} + \binom{\frac{n+k-2}{2}}{k} \right] x^k.$$

Again, using the combinatorial interpretation, the following identities are easily obtained.

Identity 3 *For $n \geq 1$,*

$$U_n(x) = T_n(x) + xU_{n-1}(x).$$

Identity 4 *For $n \geq 2$,*

$$2T_n(x) = U_n(x) - U_{n-2}(x).$$

Identity 5 *For* $n \geq 0$,

$$U_n(x) = \sum_{j=0}^{n} x^j T_{n-j}(x).$$

Identity 6 *For* $n \geq 2$,

$$T_n(x) = -U_{n-2}(x) + \sum_{j=1}^{n} x^j T_{n-j}(x).$$

With a little more creative thinking, one can also obtain the following identities.

Identity 7 *For* $n \geq 2$,

$$T_n(x) = T_{n-2}(x) + 2(x^2 - 1)U_{n-2)}(x).$$

After introducing the technique of "tailswapping," we can derive the following beautiful identities.

Identity 8 *For* $0 \leq m \leq n$,

$$2T_n(x)T_m(x) = T_{n+m}(x) + T_{n-m}(x).$$

Identity 9 *For* $0 \leq m \leq n$,

$$T_n^2(x) + T_m^2(x) = 1 + T_{n+m}(x)T_{n-m}(x).$$

Identity 10 *For* $n \geq 0$,

$$(1 + x)U_n(x) = 1 + T_{n+1}(x) + \sum_{i=1}^{n} 2T_i(x).$$

Perhaps the most beautiful (and most challenging to prove) identity is the simple composition identity.

Identity 11 *For all* $m, n \geq 0$,

$$T_m(T_n(x)) = T_{mn}(x).$$

For Chebyshev polynomials of the second kind, the identity is almost as nice.

Identity 12 *For all* $m, n \geq 0$,

$$U_{m-1}(T_n(x))U_{n-1}(x) = U_{mn-1}(x).$$