The q-Bernstein basis as a q-binomial distribution

Charalambos A. Charalambides

Department of Mathematics, University of Athens,

GR-15784 Athens, Greece

E-mail address: ccharal@math.uoa.gr

1 Introduction and summary

Let x be a real number, q a positive real number, with $q \neq 1$, and k an integer. The number $[x]_q = (1-q^x)/(1-q)$ is called *q-number* and in particular $[k]_q$ is called *q-integer*. The kth order factorial of the q-number $[x]_q$, which is defined by

$$[x]_{k,q} = [x]_q[x-1]_q \cdots [x-k+1]_q, \quad k = 0, 1, \dots ,$$

is called *q-factorial of x of order k*. The *q-binomial coefficient* (or *Gaussian polynomial*) is defined by

$$\begin{bmatrix} x \\ k \end{bmatrix}_q = \frac{[x]_{k,q}}{[k]_q!}, \quad k = 0, 1, \dots .$$

Phillips (1997) introduced the *q-Bernstein polynomial of order n* for any continuous function $f(t)$ on the interval $[0,1]$ by

$$B_n(f(t), q; x) = \sum_{r=0}^{n} f(t_r) \begin{bmatrix} n \\ r \end{bmatrix}_q x^r \prod_{i=1}^{n-r} (1 - xq^{i-1}), \quad n = 1, 2, \dots , \tag{1.1}$$

for $0 < x < 1$ and $q > 0$, with $t_r = [r]_q/[n]_q$, $r = 0, 1, \dots, n$. In the particular case $q = 1$, it reduces to the classical *Bernstein polynomial of order n*

$$B_n(f(t); x) = \sum_{r=0}^{n} f(t_r) \binom{n}{r} x^r (1-x)^{n-r}, \; n = 1, 2, \dots ,$$

for $0 < x < 1$, with $t_r = r/n$, $r = 0, 1, \dots, n$. The sequence

$$p_{n,r}(x, q) = \begin{bmatrix} n \\ r \end{bmatrix}_q x^r \prod_{i=1}^{n-r} (1 - xq^{i-1}), \quad r = 0, 1, \dots, n, \tag{1.2}$$

for $0 < x < 1$ and $0 < q < 1$ forms a normalized totally positive basis, called *q-Bernstein basis*.

Il'inskii and Ostrovska (2002), Oruc and Tuncer (2002) and Ostrovska (2003) studied the convergence of the q-Bernstein polynomials. In Ostrovska's study of the convergence of the q-Bernstein polynomial $B_n(f(t), q; x)$, for $n \to \infty$, in the case $x = q^{-m}$ with $q > 1$ and m a positive integer, the the role of q-Bernstein basis (1.2) is played by the sequence

$$\begin{aligned} p_{n,r}(q^{-m}, q) &= \begin{bmatrix} n \\ r \end{bmatrix}_q q^{-rm} \prod_{i=1}^{n-r} (1 - q^{-m+i-1}) \\ &= \begin{bmatrix} n \\ r \end{bmatrix}_{q^{-1}} q^{-r(m-n+r)} (1 - q^{-1})^{n-r} [m]_{n-r,q^{-1}}, \quad r = 0, 1, \ldots, n. \end{aligned} \tag{1.3}$$

In the present paper, the sequences (1.2) and (1.3) are shown to be the probability (mass) functions of two q-binomial distributions. Specifically, a stochastic model is introduced by considering a sequence of Bernoulli trials with probability of failure at any trial increasing geometrically with the number of previous failures. The probability function of the number of successes in n trials is derived as (1.2). In a modification of this model, in which the probability failure at any trial decreases geometrically with the number of previous failures, the probability function of the number of successes in n trials is obtained as (1.3). Further, the q-factorial and the usual factorial moments of these q-binomial distributions are derived and their limiting distributions are given. Also, using the expression of the q-moments, the q-Bernstein polynomial $B_n(f_r(t), q; x)$, for $f_r(t)$ a polynomial of degree r, is expressed as a sum of $f_m(x)$, with $m = min\{n, r\}$, and a polynomial in $1/[n]_q$ of degree at most $r - 1$, lacking constant term, with coefficients polynomials in x of degree at most m that do not depend on $[n]_q$.

References

Blomqvist, N. (1952) On an exhaustion process, *Skandinavisk Aktuarictidskrift*, **35**, 201-210.

Charalambides, Ch. A. (2002) *Enumerative Combinatorics*, Chapman & Hall/CRC, Boca Raton, FL.

Charalambides, Ch. A. and Papadatos, N. (2005) The q-factorial moments of discrete q-distributions and a characterization of the Euler distribution, in *Advances on models, characterizations and applications* (Eds. N. Balakrishnan, I. G. Bairamov and O. L. Gebizlioglu), pp. 57-71.

Dunkl, C. F. (1981) The absorption distribution and the q-binomial theorem, *Communications in Statistics-Theory and Methods*, **10**, 1915-1920.

Gould, H. W. (1958) A theorem concerning the Bernstein polynomials, *Mathematics Magazine*, **31**, 259-264.

Il'inskii, A. and Ostrovska, S. (2002) Convergence of generalized Bernstein polynomials, *Journal of Approximation Theory*, **116**, 100-112.

Jing, S. G. (1994) The q-deformed binomial distribution and its asymptotic behaviour, *Journal of Physics A: Mathematical and General*, **27**, 493-499.

Kemp, A. (1998) Absorption sampling and the absorption distribution, *Journal of Applied probability*, **35**, 489-494.

Oruc, H. and Tuncer, N. (2002) On the convergence and iterates of q-Bernstein polynomials, *Journal of Approximation Theory*, **117**, 301-313.

Ostrovska, S. (2003) q-Bernstein polynomials and their iterates, *Journal of Approximation Theory*, **123**, 232-255.

Philips, G. M. (1997) Bernstein polynomials based on q-integers, *Annals of Numerical Analysis*, **4**, 511-518.