The Euler characteristic of some lattices of paths *

Luca Ferrari † Emanuele Munarini ‡

1 Preliminary notations and definitions

Let Γ be a finite subset of $\mathbb{Z}$. A Γ*-path of length* n is a function $f : [0, n] = \{0, 1, \ldots, n\} \to \mathbb{N}$ such that $f(0) = f(n) = 0$ and $f(k + 1) - f(k) \in \Gamma$, for every $k \in [0, n - 1]$. This definition is equivalent to the usual one, which asserts that a Γ-path of length n is a sequence of n steps of the type $(1, k)$, with $k \in \Gamma$, starting from the origin, ending on the x-axis and never going below that axis. Γ-paths of length n, considered as functions, can be ordered coordinate-wise, by declaring $f \leq g$ whenever $f(k) \leq g(k)$, for every $k \in [0, n]$. In this way we obtain a poset $\mathcal{C}_n^\Gamma$, which is not necessarily a lattice. In [**?**] some conditions for $\mathcal{C}_n^\Gamma$ to be a distributive lattice are given. In particular, it is shown that $\mathcal{C}_n^\Gamma$ is a distributive lattice if $\Gamma = \{-b, a\}$, for $a, b \in \mathbb{N}$. We will denote $\mathcal{D}_n^{(a,b)}$ the lattice of $\{-b, a\}$-paths, and their elements will be called *Dyck-like paths of type* (a, b). These classes of paths were previously considered, for instance, in [**?**]

2 Dyck-like paths: valuations and characteristic

A *valuation* of a distributive lattice D with values in the real field $\mathbb{R}$ is a function $\nu : D \to \mathbb{R}$ such that $\nu(x \vee y) + \nu(x \wedge y) = \nu(x) + \nu(y)$ for every $x, y \in D$. A valuation of a finite distributive lattice D is uniquely determined by the values it takes on the set of the join-irreducibles of D, and these values can be arbitrarily assigned [**?**]. For any valuation ν, the following generalized form of the principle of inclusion-exclusion holds:

$$\nu(x_1 \vee \cdots \vee x_n) = \sum_{\substack{S \subseteq [n] \\ S \neq \emptyset}} (-1)^{|S|-1} \nu \left(\bigwedge_{i \in S} x_i \right). \tag{1}$$

The *(Euler) characteristic* of D is the unique valuation χ such that $\chi(\widehat{0}) = 0$ and $\chi(x) = 1$ for each join-irreducible x of D (where $\widehat{0}$ denotes the minimum of a lattice).

2.1 The characteristic of a Dyck-like lattice

A *Dyck-like lattice* is a distributive lattice whose spectrum (i.e. the poset induced by its join-irreducibles) is a ranked poset such that there is a labelling of its elements using positive integers with the following property: every antichain $S = \{s_1, \ldots, s_n\}$ of join-irreducibles can be linearly

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†Dipartimento di Sistemi e Informatica, viale Morgagni 65, 50134 Firenze, Italy ferrari@dsi.unifi.it
‡Politecnico di Milano, Dipartimento di Matematica, Piazza Leonardo da Vinci 32, 20133 Milano, Italy emanuele.munarini@polimi.it

ordered so that the labels of the elements of S are distinct and, if s_1 and s_n are the elements having minimum and maximum labels, respectively, then $s_1 \wedge s_n = s_1 \wedge s_2 \wedge \cdots \wedge s_{n-1} \wedge s_n$.

Proposition 2.1 *For any $a, b \in \mathbb{N}$, the lattice $\mathcal{D}_n^{(a,b)}$ of Dyck-like paths of length $n \cdot \ell(a,b) = n \cdot \frac{a+b}{\gcd(a,b)}$ is a Dyck-like lattice, for every $n \in \mathbb{N}$.*

The following assertion, a sort of converse of the previous proposition, seems to be plausible:

Conjecture. Every finite Dyck-like lattice can be represented as a sublattice of a lattice of Dyck-like paths of suitable length.

An element x of a distributive lattice D is called *quasi-join-irreducible* when there exists an antichain $s_1, \ldots, s_k$ of join-irreducibles such that $x = s_1 \vee \cdots \vee s_k$ and $s_i \wedge s_{i+1} \neq \widehat{0}$, for every $i < k$.

Proposition 2.2 *Let D be a Dyck-like lattice such that the meet of any two join-irreducibles is $\widehat{0}$ or a join-irreducible. Then every quasi-join-irreducible element has characteristic equal to 1.*

An element x of a finite distributive lattice is said to have a *quasi-join-irreducible decomposition* when it can be expressed as a join of quasi-join-irreducible elements $x_1, \ldots, x_k$ such that $x_i \wedge x_j = \widehat{0}$, for every $i \neq j$.

Proposition 2.3 *Every element of a Dyck-like lattice has a quasi-join-irreducible decomposition.*

Theorem 2.1 *Let D be a Dyck-like lattice such that the meet of any two join-irreducibles is $\widehat{0}$ or a join-irreducible. Then, for every $x \in D$, $\chi(x)$ is the number of quasi-join-irreducibles in a decomposition of x.*

Corollary 2.1 *In the above hypotheses, two quasi-join-irreducible decompositions of the same element x have the same number of elements.*

For the lattices $\mathcal{D}_n$ of Dyck paths it is possible to give a combinatorial interpretation of the characteristic. To do this, we need one more definition. A *tunnel* of a Dyck path [**?**, **?**] is a pair (u, d) constituted by an up step u and the down step d which can be reached by drawing an horizontal line starting from u and going to the right of u. A tunnel has *height* k when the step u of the tunnel starts at height k. A tunnel of height k will be also called a *k-tunnel*.

Proposition 2.4 *A Dyck path $x \in \mathcal{D}_n$ is quasi-join-irreducible if and only if it has precisely one tunnel of height 1.*

Proposition 2.5 *The meet of two join-irreducibles of $\mathcal{D}_n$ is $\widehat{0}$ or a join-irreducible. More precisely, $\{\widehat{0}\} \cup \mathbf{Spec}(\mathcal{D}_n)$ is a ranked meet-semilattice, where the rank of a path is equal to the maximum height of its tunnels (i.e. $r(x) = k$ whenever x has a k-tunnel but not a $(k+1)$-tunnel).*

Corollary 2.2 *The characteristic of a Dyck path $x \in \mathcal{D}_n$ is the number of its 1-tunnels.*

2.2 Generalized characteristics of a Dyck-like lattice

In this section we always suppose that D is a finite Dyck-like lattice, P is its spectrum and $\widehat{P} = \{\widehat{0}\} \cup P$ is a ranked meet-semilattice with rank function $r_{\widehat{P}}$. Given $k \in \mathbb{N}$, we define the valuation χ_k on D by setting $\chi_k(x) = 1$ when $r_{\widehat{P}}(x) \geq k$ and $\chi_k(x) = 0$ when $r_{\widehat{P}}(x) < k$, for every join-irreducible x. In particular $\chi_1 = \chi$. Our aim is to evaluate $\chi_k(x)$ for every $x \in D$.

Proposition 2.6 *Let $x = x_1 \vee \cdots \vee x_m$ where $m \geq 1$, every x_i is a join-irreducible and $r_{\widehat{P}}(x_i \wedge x_{i+1}) \geq k$, for $i < m$. Then $\chi_k(x) = 1$.*

Proposition 2.7 *Let $x = x_1 \vee \cdots \vee x_m$, where $m \geq 1$, $x_i = x_{i,1} \vee \cdots \vee x_{i,m_i}$ is a join of join-irreducibles such that $r_{\widehat{P}}(x_{i,\alpha_m} \wedge x_{i,\alpha_m+1}) \geq k$, for every $i = 1, \ldots, m$, and $r_{\widehat{P}}(x_i \wedge x_j) < k$ for every $i \neq j$. Then $\chi_k(x) = m$.*

Proposition 2.8 *Every element x of D can be written as $x = x_1 \vee \cdots \vee x_h \vee x_{h+1} \vee \cdots \vee x_m$ where (i) for every $i = 1, \ldots, h$, $x_i = x_{i,1} \vee \cdots \vee x_{i,m_i}$ is a join of join-irreducibles with $r_{\widehat{P}}(x_{i,\alpha_m} \wedge x_{i,\alpha_m+1}) \geq k$ and $r_{\widehat{P}}(x_i \wedge x_j) < k$ for every $i \neq j$, and (ii) for every $i = h+1, \ldots, m$, x_i is a join-irreducible with $r_{\widehat{P}}(x_i) < k$.*

We will call a *k-quasi-join-irreducible decomposition* of an element $x \in D$ any decomposition of the kind described in the previous proposition. Moreover, using the above notations, we will call *k-quasi-join-irreducible* all the elements x_i with $i = 1, \ldots, h$.

Theorem 2.2 *For every element $x \in D$, $\chi_k(x)$ is equal to the number of k-quasi-join-irreducibles in any k-quasi-join-irreducible decomposition of x.*

As a consequence of this theorem, we have that the number of k-quasi-join-irreducibles in any k-quasi-join-irreducible decomposition of an element x is constant. Finally, from theorem **??** and proposition **??** the desired interpretation of the generalized characteristics of Dyck lattices follows.

Proposition 2.9 *For every Dyck path $x \in \mathcal{D}_k$, $\chi_k(x)$ is the number of k-tunnels of x.*

3 The characteristic of Motzkin lattices

The case of Motzkin paths is quite different from the Dyck one. Indeed, Motzkin paths are not Dyck-like paths even though the lattice $\mathcal{M}_n$ of Motzkin paths of length n is a Dyck-like lattice. Notice that $\mathcal{M}_n$ is isomorphic to the lattice of Dyck paths of length $2n$ with at most two consecutive down steps [**?**]. This perfectly agrees with our previous conjecture on the representation of Dyck-like lattices.

To give a combinatorial description of the characteristic for Motzkin lattices we can not use the theory developed in the previous sections, since in $\mathcal{M}_n$ it could happen that the meet of two join-irreducibles is neither $\widehat{0}$ nor a join-irreducible. For instance, the meet of the paths u^2d^2h and hu^2d^2 in $\mathcal{M}_5$ is $huhdh$, which is clearly not join-irreducible. However, we can prove a result similar to Proposition **??** for the meet of two join-irreducibles in $\mathcal{M}_n$. To state such a result, we need the following definition. A *truncated pyramid* of *height* k and *length* m is any sequence of $k \geq 1$ up steps followed by a sequence of m horizontal steps followed by a sequence of k down steps, i.e. $u^k h^m d^k$. Let $T_{n,m,k}$ be the set of Motzkin paths of length n with a unique truncated pyramid of length m and height k, i.e. Motzkin paths with only horizontal steps at height 0, except for a unique truncated pyramid of height k and length m.

Now we are ready to state our result on the meet of two join-irreducible Motzkin paths.

Proposition 3.1 *In $\mathcal{M}_n$ the meet of two join-irreducibles which is not $\widehat{0}$ is either a join-irreducible or an element of $T_{n,1,k}$ (i.e., a Motzkin path with a unique truncated pyramid of length 1).*

Hence the paths in $T_{n,1,k}$ are the unique Motzkin paths which are not join-irreducibles and can be obtained as a meet of two join-irreducibles. Having this in mind, we can try to replicate the arguments developed for Dyck paths in order to get analogous results in the Motzkin case.

Lemma 3.1 *Let x be a Motzkin path with a unique truncated pyramid of height h and length m. Then $\chi(x) = (-1)^{h+1}m + 1$.*

Now we are ready to state the fundamental step in the determination (and combinatorial interpretation) of the characteristic of $\mathcal{M}_n$.

Proposition 3.2 *Let $x \in \mathcal{M}_n$ be a quasi-join-irreducible having o_x horizontal steps at odd height and e_x horizontal steps at even nonzero height. Then $\chi(x) = o_x - e_x + 1$.*

It is clear that every element of a Motzkin lattice has a quasi-join-irreducible decomposition: this follows from proposition **??**, since Motzkin lattices are Dyck-like lattices.

Theorem 3.1 *For every Motzkin path x, $\chi(x) = \|x\| + o_x - e_x$, where $\|x\|$ denotes the number of quasi-join-irreducibles in a decomposition of x.*

This theorem can be translated into combinatorial terms as follows. Let $f_h = f_h(x)$ be the number of *flats* (i.e., sequences of consecutive horizontal steps) of x at height h; $p_h = p_h(x)$ be the number of peaks of x at height h; and $t_h = t_h(x)$ be the number of tunnels of x at height h.

Theorem 3.2 *For every Motzkin path x, $\chi(x) = o_x - e_x + f_1 + p_1 + t_1$.*

4 The characteristic of Schröder lattices

The results of section **??** can be used to give a combinatorial interpretation also for the characteristic of the lattices generated by Schröder paths. Specifically we have

Proposition 4.1 *The Schröder lattices $\mathcal{S}_n$ are Dyck-like lattices such that the meet of any two join-irreducibles is $\widehat{0}$ or a join-irreducible.*

Theorem 4.1 *A Schröder path $x \in \mathcal{S}_n$ is quasi-join-irreducible if and only if it has precisely one tunnel of height 0. So, the characteristic of a Schröder path is given by the number of its 0-tunnels.*

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