Enumeration of Osculating Lattice Paths

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The problem of enumerating osculating lattice paths has been well-studied, and many proofs exist for the enumeration of pairs of such paths. The case with three paths is much more difficult; however Bousquet-Mélou has given a proof using functional equations and the obstinate kernel method. No method previously known generalises to one for N paths. In this paper, we describe a method of enumerating osculating lattice paths using an involution.

We consider **binomial paths** as sequences of vertices $p = v_0 v_1 \dots v_t$ such that $v_i \in \mathbb{Z} \times \mathbb{Z}$ and $(v_{i+1} - v_i) \in \{(0,1),(1,0)\}$. We are concerned with the enumeration of sets of N paths $\mathbf{p}$ on the integer lattice which are **osculating**, namely if whenever two paths $p_i, p_j \in \mathbf{p}$ $(i < j)$ share a vertex (x,y) then $(x-1,y)$, $(x,y+1) \in p_i$, and $(x,y-1),(x+1,y) \in p_j$. Such a shared vertex is called an *osculation*. We will usually consider *weighted* sets of osculating paths where $w(\mathbf{p}) = \omega^{\#\,\text{osculations}}$.

Since we will use an involution, this requires a much larger set within which the desired set resides. We augment the set of sets of binomial paths by introducing markings, which occur on pairs of edges after crossings or osculations. We require that if any intersection is marked then all previous intersections between those paths are also marked, and only the last marked intersection may be a crossing, the others *must* be osculations.

1 Involution for N Paths

Since marked paths are not easily enumerable, objects called *stacks* are introduced to obtain formulae from the augmented set. The involution then has several steps. A stack is progressively swept out (akin to a seismograph) into a set of paths, however the algorithm may halt before the entire stack has been used. In fact, the fixed point set corresponds to those stacks which can be swept out completely; the remaining pairs of stacks and paths undergo a simple swapping process. A backward sweep follows, and the entire process gives an involution on the stacks. To ensure the symobols used in our results are defined, we provide the following technical definitions:

Definition 1 (Indexed Inversion Table) *An* indexed inversion table *for a permutation σ is an inversion table where the entries are weak compositions of k_i, the ith entry each having precisely a_i components. Each component is signed (by convention, zero is always signed $-$) and we write $I^\star(\sigma) = (\mathbf{k}_1, \mathbf{k}_2, \ldots, \emptyset)$ where $\mathbf{k}_i = (k^{\pm}_{ij_1}, k^{\pm}_{ij_2}, \ldots, k^{\pm}_{ij_{a_i}})$.*

An indexed inversion table provides a weighting for a permutation, such that each inversion $i < j : \sigma_j < \sigma_i$ is weighted by $k^{\pm}_{ij}$, a component of the ith entry in the inversion table. These can be represented pictorially by labelling the intersections in a Hasse diagram, with $k^{\pm}_{ij}$ labelling the intersection between lines i and j.

Definition 2 (Marked Path Stack) *Given endpoints* **a** *and* **b***, these define a permutation σ. For some indexed inversion table $I^\star(\sigma)$, we have a class of* marked path stacks *defined by:*
For each $i \in [N]$:

1. *Let $u_i = u \cdots u$ where*

$$|u_i| = \sum_{j>i} k^{\pm}_{ij} + \#\{k^{+}_{ji} : j < i\} - \#\{k^{+}_{ij} : j > i\}.$$

2. *Let $r_i = r \cdots r$ where*

$$|r_i| = \sum_{j<i} k^{\pm}_{ji} + \#\{k^{+}_{ij} : j > i\} - \#\{k^{+}_{ji} : j < i\}.$$

3. *Finally, f_i is a free arrangement of $b_i - a_i - |u_i|$ u's, and $t - b_i + a_i - |r_i|$ r's.*

These are restricted further as a result of the number of "up" and "right" steps. Finally, we weight a marked path stack $\bar{\mathbf{s}}$ by

$$w(\bar{\mathbf{s}}) = (-1)^{|\mathcal{I}_\sigma| + \#\{k^{+}_{ij} : i<j\}} \omega^k$$

where $k = \sum_{i<j} k^{\pm}_{ij}$.

1.1 Involution for General N

The forward sweep algorithm essentially deals with one of several cases at each iteration:

1. the stack is empty;
2. there are collisions and marked steps may be used;
3. there are collisions but insufficient markings;
4. one of the paths has not collided, but its free portion of the stack has been exhausted; and
5. there are no collisions and the algorithm may continue.

The first of these produces the output for a fixed point, the next three for those elements upon which the involution operates.

Theorem 1 *Let $\mathcal{O}_t^\star$ be the set of all configurations of N osculating paths of length t with the ith path starting at $(-a_i, a_i)$ and ending at $(t-b_i, b_i)$, where there is at least one osculation. Then*

$$\sum_{\mathbf{p}\in\mathcal{O}_t^\star} w(\mathbf{p}) = \sum_{k\geq 1}\sum_{\sigma\neq 1}\sum_{I^\star(\sigma)}\prod_{i=1}^{N}\omega^{\sum_{i<j}k_{ij}^{\pm}}(-1)^{|\mathcal{I}_\sigma|+\#\{k_{ji}^{+}:j<i\}} \times\binom{t-\sum_{i\neq j}k_{ij}^{\pm}}{b_{\sigma_i}-a_i-\sum_{j>i}k_{ij}^{\pm}-\#\{k_{ji}^{+}:j<i\}+\#\{k_{ij}^{+}:j>i\}}.$$

2 Connections to Other Results

Clearly the formula obtained by Corollary **??** is impractical for large numbers of paths, as the number of terms in the expression grows superexponentially in the number of paths. Accordingly, we seek to find a more useful expression. Using the constant term method, a variation of the *Bethe Ansatz*, we obtain the solution from a Laurent Series expansion. This is more computationally expensive for each term, however the summation always has $(N!-1)$ terms, where there are N paths. For very large N this again becomes computationally impractical, but there is no product form for general osculating paths.

2.1 Alternating Sign Matrices

Corollary 2 *The* Bethe Ansatz *for alternating sign matrices is given by*

$$|\mathcal{A}^N| = 1 + CT\left[\sum_{\sigma\neq 1}\prod_{i=1}^{N}\lambda_i^{2N-(i+\sigma_i)+1}z_i^{3i+3-\sigma_i}\prod_{(ij)\in\mathcal{I}_\sigma}\left(-\frac{\lambda_i\lambda_j - z_j\bar{z}_i}{\lambda_i\lambda_j - z_i\bar{z}_j}\right)\right]$$

where $\lambda_i = z_i + \bar{z}_i$ and $\bar{z}_i = 1/z_i$.

For various symmetry classes of alternating sign matrices we can also determine Bethe Ansatz results. These make use of path bijections analogous to that for general alternating sign matrices.