History and a Survey of Lattice Path Enumeration

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In celebration of the 6th International Conference on Lattice Path Counting and Applications, it is befitting to overview the history of lattice path enumeration and to record how the topic has progressed thus far. Particularly for the conference talk we wish to expand and expel the history of André's Reflection Principle.

It is often thought that the first lattice path enumeration problem is a translation of the ballot theorem posed by Bertrand in 1887 and first solved by André using the reflection principle. Takács points out in 1967 that the ballot problem is equivalent to a game of chance discussed in 1708 by DeMoivre. André solves the ballot problem using a clever approach that did not include a lattice path application. The use or invention of André's reflection principle can not be found in André's work. So what is the ballot problem and what is André's proof? What is Andre's Reflection Principle? Who had the idea for it? Did it start with the physicists? Who took that first step and applied a reflective bijection to lattice paths? We present evidence to the history of Andre's reflection principle starting with Lord Kelvin's method of images in 1843 and include early physicists studying Brownian motion along with some early lattice path works.

We find lattice path techniques as early as 1878 in Whitworth to help picture a combinatorial problem, but it is not until the early 1960's that we find lattice path enumeration presented as a mathematical topic on its own. The number of papers pertaining to lattice path enumeration has more than doubled each decade since 1960. We summarize the types of lattice path problems and solution methods we found in the literature in the last 25 years. Lattice paths have been studied mostly on the integer rectangular lattice using a two-element step set, either up and to the right $\{\uparrow, \longrightarrow\}$ or diagonally up and down $\{\nearrow, \searrow\}$. Also studied are three-element step sets, primarily for either $\{\uparrow, \longrightarrow, \nearrow\}$ or $\{\nearrow, \searrow, \longrightarrow\}$. Boundaries are predominately the diagonal or parallels to it. Less prevalent have been lattice path enumeration problems on nonrectangular lattices, or those using infinite step sets, or lower boundary lines with rational slopes. Methods found are combinatorial; analytical, like the kernel method and probabilistic methods; and algebraic, as in Riordan arrays, Gessel -Viennot method, and finite operator calculus. Bijections to lattice paths connect results to other mathematical objects including other lattice paths, Young Tableau and compositions via vector representation. Some of this work appears in the author's 2005 dissertation.