Multilinear Generating Function for Charlier Polynomials

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For the last few decades, combinatorial methods have been effectively used in the study of orthogonal polynomials such as Hermite polynomials, Charlier polynomials, Jacobi polynomials, etc. This talk will focus on the Charlier polynomials which are usually defined by the following formula:

$$c_n(a,r) = {}_2F_0(-n,-a;-;-r^{-1})$$

where ${}_2F_0(a,b,c;z) = \sum_{n\geq 0} \frac{(a)_n\,(b)_n}{(c)_n}\frac{z^n}{n\,!}$, is the usual hypergeometric series. For the purpose of combinatorial interpretation, we will normalize them as follows:

$$C_n(a,r) = r^n c_n(a,r) = r^n\,{}_2F_0(-n,a;-;-r^{-1}) = \sum_{k=0}^{n}\binom{n}{k}(a)_k r^{n-k}\,.$$

With this normalization, we show that the Charlier polynomial $C_n(a,r)$ is the generating polynomial for certain directed graphs on n vertices, called the Charlier configurations. If $[n]$ denotes the set $\{1,2,\ldots,n\}$, then a Charlier configuration on $[n]$ is a pair $\Phi = ((A,\sigma),B)$, where (A,B) is an ordered partition of $[n]$ and σ is a permutation of A. The configuration Φ can be represented by a directed graph or digraph with vertex set $[n]$ and with an edge going from i to j if and only if $\sigma(i)=j$. By assigning a weight a to each cycle of σ and a weight r to each point of B, the Charlier polynomial $C_n(a,r)$ can be shown to be the generating polynomial for all Charlier configurations on $[n]$.

With this combinatorial interpretation, we prove the following multilinear formula:

$$(1)\quad \sum_{(n_{ij})} \frac{\prod\limits_{1\leq i<j\leq k} {x_{ij}}^{n_{ij}}}{\prod\limits_{1\leq i<j\leq k} n_{ij}!} C_{n_1}(a_1,r_1)\cdots C_{n_k}(a_k,r_k)$$

$$= \prod_{1\leq i<j\leq k} e^{r_i r_j x_{ij}} \sum_{(n_{ij})} \prod_{1\leq i\leq k} \frac{(a_i)_{n_i}}{\left(1-\sum_{j\neq i} r_j x_{ij}\right)^{n_i+a_i}} \frac{\prod\limits_{1\leq i<j\leq k} {x_{ij}}^{n_{ij}}}{\prod\limits_{1\leq i<j\leq k} n_{ij}!}\,,$$

where each sum runs over all $k\times k$ symmetric matrices (n_{ij}) with non-negative integral entries and with diagonal entries zero, and $n_i = \sum_{j=1}^{k} n_{ij}$ for $1\leq i\leq k$. In the combinatorial proof of the formula, we show that the two sides of the formula enumerate the same set of directed graphs which are obtained by superimposing k Charlier configurations on the same set of n vertices with certain restrictions on the vertices, where $n=\sum n_{ij}$.

I will end the talk with some special cases of the multilinear formula such as the bilinear formula with pairs of Charlier polynomials, and a formula with derangements. This is joint work with Ira Gessel.