

On q -Discrete and q -Continuous Distributions Based on q - Orthogonal Polynomials

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Abstract

Let ν be a probability measure in R with finite moments of all orders

$$s_m = \int_{x \in R} x^m d\nu(x).$$

Then it is well known [12] that there exists a sequence of normalized orthogonal polynomials $\{P_m(x)\}$ with respect to the measure ν satisfying the recurrence relation

$$\begin{aligned} xp_m(x) &= p_{m+1}(x) + a_m p_m(x) + b_m p_{m-1}(x), \\ m &\geq 1, \end{aligned} \tag{1}$$

with initial conditions

$$p_0(x) = 1, \quad p_1(x) = x - a_0.$$

Moreover, the polynomials $\{P_m(x)\}$ satisfy the orthogonality relation

$$\int_{x \in R} P_m(x) P_\nu(x) d\nu(x) = \delta_{m,\nu} b_1 b_2 \cdots b_m, \quad m, \nu \geq 0 \tag{2}$$

where $\delta_{m,\nu}$ the Kronecker delta.

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The polynomials $\{P_m(x)\}$ depend on the moment sequence $\{s_m\}_{m \geq 0}$ and they can be obtained from the formula

$$P_m(x) = \sqrt{\frac{b_1 b_2 \cdots b_m}{D_{m-1} D_m}} \begin{vmatrix} s_0 & s_1 & \cdots & s_m \\ s_1 & s_2 & \cdots & s_{m+1} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ s_{m-1} & s_m & \cdots & s_{2m-1} \\ 1 & x & \cdots & x^m \end{vmatrix}, \quad (3)$$

where $D_m = \det(\{s_{i+j}\}_{0 \leq i, j \leq m})$ denotes the Hankel determinant. Conversely, Favard's theorem ensures the existence of a probability measure ν on R for which the sequence of polynomials determined by the recurrence relation (1) are orthogonal. The mean value and the variance of the random variable X in R with probability density function $\nu(x)$ are given respectively by

$$\mu = a_0 \quad \text{and} \quad \sigma^2 = b_1.$$

If $a_m = 0$ then all moments of odd order are zero

$$s_{2m+1} = \int_{x \in R} x^{2m+1} d\nu(x) = 0$$

(see Saitoh and Yoshida [10], Christiansen [2]).

Saitoh and Yoshida [10] introduced certain q -deformations of the binomial and Poisson distributions in quantum probability space by virtue of q -deformed sequences of orthogonal polynomials. Moreover, they have studied the asymptotic behaviour by showing that are compatible with a standardized Stieltjes-Wigert distribution and a standardized q -continuous q -Gauss distribution, respectively.

Kyriakoussis and Vamvakari [8] defined for $0 < q < 1$ two deformed q -binomial distributions induced by the normalized q -Krawtchouk and the normalized dual q -Krawtchouk orthogonal polynomials respectively. Interpretations of these q -discrete probability distributions had been provided by considering, for suitable values of q , multi-dimensional vector spaces over the finite field of $1/q$ and $1/q^2$ elements respectively. Furthermore, they have studied their asymptotic behaviour by showing that are compatible with a

standardized Stieltjes-Wigert distribution and a standardized q -continuous q -Gauss type distribution, respectively

Various q -analogues for $0 < q < 1$, of the classical binomial and Poisson distributions have been studied by many authors. Among them we refer to Kemp [3], [4] and [5], Sicong [11], Kupersmidt [6] and Charalambides [1]. Van Leeuwen and Maassen [9] studied a q -deformed standardized gaussian distribution, which takes the semicircle distribution for $q = 0$ and recovers the standardized Gaussian distribution for $q = 1$. This q -deformed Gaussian distribution is associated with the continuous q -Hermite orthogonal polynomials.

Kyriakoussis and Vamvakari [7], introduced the q -analogue for $0 < q < 1$, of continuous distributions. Moreover, they have defined a q -Gauss distribution based on discrete q -Hermite of type I orthogonal polynomials and the corresponding standardized q -Gauss distribution for $0 < q < 1$ which reduces to the standardized Gauss when $q \rightarrow 1$.

In this talk, we present for $0 < q < 1$ the q -analogue for $0 < q < 1$, of discrete and continuous distributions based on discrete q -orthogonal polynomials. Analytically, we consider the q -discrete distributions induced by discrete q -orthogonal polynomials satisfying the orthogonality relation (2) in which the integral $\int(\cdot)dx$ is replaced by the q -sum $\sum_{x \in S}(\cdot)$ with S a discrete spectrum based on the points q^{-k} or $[k]_q$, $k = 0, 1, \dots, n$. Also, we consider the q -continuous distributions by virtue of discrete q -orthogonal polynomials in which the integral $\int(\cdot)dx$ of (2) is replaced by the q -integral $\int(\cdot)d_qx$. Moreover, we present the two types of q -binomial distributions induced by the normalized q -Krawtchouk and the normalized dual q -Krawtchouk orthogonal polynomials respectively introduced in [8]. In addition, we study the q -deformed orthogonal polynomials provided by the classical q -binomial and q -negative binomial distributions defined in [3] and [4]. Furthermore, we present the q -Gauss distribution based on discrete q -Hermite of type I orthogonal polynomials and the corresponding standardized one introduced in [7]. Finally, we define a q -deformed Gamma distribution by which a type of discrete q -Laguerre orthogonal polynomials is provided.

References

- [1] Ch. A. Charalambides, Moments of a class of discrete q -distributions, *J. Stat. Plan. Infer.* **135** (2005), 64–76.
- [2] J.S. Christiansen, Indeterminate moment problems within the Askey-scheme, PH.D. thesis, Institute for Mathematical Sciences, University of Copenhagen, 2004.
- [3] A.W. Kemp and C.D. Kemp, Welson’s dice revisited, *The American Statistician*, bf 45, 216-222.
- [4] A.W. Kemp, Steady state Markov chain models for the heine and Euler distributions, *J. Appl. Probab.* **29** (1992), 869-876.
- [5] A.W. Kemp, Certain q -analogue of the binomial distribution, *Sankhy a: The Indian Journal of Statistics* **64** (2002), 293-305.
- [6] B.A. Kupershmidt, q -Probability: I. Basic Discrete Distributions, *Journal of Nonlinear Math. Phys.* **7** (No.1) (2000), 73-93.
- [7] A. Kyriakoussis and M.G. Vamvakari, q -analogue of continuous distributions—A q -Gauss distribution based on discrete q -Hermite of type I orthogonal polynomials (submitted).
- [8] A. Kyriakoussis and M.G. Vamvakari, On q -binomial type Distributions based on q -Krawtchouk Orthogonal Polynomials—Asymptotic Behaviour (submitted).
- [9] H. Van Leeuwen and H. Maassen, A q -deformation of the Gauss distribution, *J. Math. Phys.* **36** (1995), 4743–4756.
- [10] A. Saitoh and H. Yoshida, A q -deformed Poisson distribution based on orthogonal polynomials, *J. Phys. A: Math. Gen.* **33**, (2000), 1435-1444.
- [11] J. Sicong, The q -deformed binomial distribution and its asymptotic behaviour, *J. Phys. A: Math. Gen.* **27** (1994), 493-499.
- [12] G. Szëgo, *Orthogonal Polynomials*, AMS Coll. Publ. vol. **23**, Providence, RI:American Mathematical Society, 1939.