Pattern Avoiding Dyck Paths, and Finite Operator Calculus

Part I: Patterns of Length Four

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The generating functions for Dyck paths avoiding any of the sixteen patterns made from combinations of four up and down steps are known *(A. Sapounakis, I. Tasoulas, P. Tsikouras, 2007)*; however, these generating functions enumerate the number of paths returning to the x-axis, and therefore many patterns are indistinguishable, giving the same counts like $uddu$ and $duud$, where we write a u for an up step and a d for a down step. We will call a Dyck path any lattice path with up and down steps that stay weakly above the x-axis and end at some point (n, m), where $n+m$ is necessarily even. We denote by $Dyck\,(n, m; p)$ the number of such paths that avoid the pattern p. Here are three examples:

m									1
7							1		
6						1		7	
5					1		6		
4				1		5		16	
3			1		4		11		
2		1		3		7		17	
1	1		2		4		9		
0	1	1		2		4		9	
	0	1	2	3	4	5	6	7	n

$Dyck\,(n, m; uddu)$

									1
							1		
						1		3	
					1		3		
				1		3		9	
			1		3		9		
		1		3		8		22	
	1		2		5		13		
1		2		5		13			
1	1		2		5		13		
0	1	2	3	4	5	6	7	8	

$Dyck\,(n, m; duuu)$

									1
								1	
							1		2
						1		2	
					1		2		
				1		2		7	
			1		2		6		
		1		2		5		13	
	1		2		4		9		
1		2		4		9			
0	1	2	3	4	5	6	7	8	

$Dyck\,(n, m; uduu)$

There are still equivalent cases; the numbers $Dyck\,(n, m; uudu)$ and $Dyck\,(n, m; uduu)$ are the same, and so are $Dyck\,(n, m; dudd)$ and $Dyck\,(n, m; ddud)$.

We use Rota's Finite Operator Calculus as a method to solve recursions; in the case of pattern avoidance, recursions in two variables, the degree and the argument of a polynomial sequence. In the first example, the polynomial sequence can easily be recognized, beginning with the constant polynomial on the diagonal, the linear polynomial on the first subdiagonal, etc. The second example obviously does not give us a polynomial sequence directly. However, differencing and rearranging the terms will lead to success. The third example actually consists of two polynomial sequences (ignoring the bottom row). There is only one such case among the 16 cases of patterns of length four. Neither of the two polynomial sequences can be expanded by the Finite Operator Calculus. The remaining 15 cases can all be understood in terms of Finite Operator Calculus. Cases that can be calculated for arbitrarily long patterns will be discussed in Part II of this series.