A Lattice Path Gallery for Fibonacci, Catalan, and Vandermonde Determinants

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Combinatorial interpretations of determinants can bring deeper understanding to their evaluations; this is especially true when the entries of a matrix have natural graph theoretic descriptions. Lindström, Gessel, and Viennot [5, 6] reveal how the determinant counts signed nonintersecting path-systems in an associated directed graph. For an $n \times n$ matrix $A = (a_{ij})$, the general idea is to create an acyclic directed graph D with n origin nodes, $o_1, o_2, \ldots, o_n$, and n destination nodes, $d_1, d_2, \ldots, d_n$, so that the number of paths from origin o_i to destination d_j is a_{ij}. Given a permutation σ in S_n, the product $\prod_{i=1}^{n} a_{i\sigma(i)}$ counts the ways to construct n directed paths in D where the i^{th} path goes from origin o_i to destination $d_{\sigma(i)}$. We call such a system of n directed paths an *n-route*. Since $\det(A) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{i\sigma(i)}$, where $\operatorname{sgn}(\sigma)$ is the sign of the permutation, the determinant is the number of n-routes induced by even permutations (called *even n-routes*) minus the number of n-routes induced by odd permutations (called *odd n-routes*). A sign reversing involution exists between even and odd n-routes provided some vertex of D is shared by two paths in the route, i.e. whenever two paths intersect. So calculating the determinant reduces to determining the number of even *nonintersecting* n-routes minus the number of odd *nonintersecting* n-routes.

Hankel matrices whose entries are either Fibonacci numbers, Catalan numbers, or sums of Catalan numbers have natural interpretations as lattice paths leading to insightful evaluations [1, 2, 3, 7, 8]. A gallery of determinants and their related directed graphs will be surveyed. Of particular interest are determinantal identities combining several of these quantities. For example, let

$$S_n^t = \begin{bmatrix} C_t + C_{t+1} & C_{t+1} + C_{t+2} & \cdots & C_{t+n-1} + C_{t+n} \\ C_{t+1} + C_{t+2} & C_{t+2} + C_{t+3} & \cdots & C_{t+n} + C_{t+n+1} \\ \vdots & \vdots & \ddots & \vdots \\ C_{t+n-1} + C_{t+n} & C_{t+n} + C_{t+n+1} & \cdots & C_{t+2n-2} + C_{t+2n-1} \end{bmatrix}.$$

where $C_t = \frac{1}{t+1}\binom{2t}{t}$, the t-th Catalan number. Using Hankel transforms and generating functions, Cvetković, Rajković, and Ivković [4] showed that $\det(S_n^0) = f_{2n}$ and $\det(S_n^1) = f_{2n+1}$ where $f_0 = 1$, $f_1 = 1$, and for $n \geq 2$, $f_n = f_{n-1} + f_{n-2}$. The simplicity of these answers begs for an elegant combinatorial solution and can be found in a bijection between nonintersecting n-routes in an associated digraph and tilings of a rectangle with squares and dominoes. In fact, this bijection can then be generalized to calculate $\det(S_n^t)$ when $t = 2$ or 3.

More creativity is required to uncover a useful lattice path interpretation of the Fibonacci determinant

$$\det \begin{bmatrix} f_{n+3} & f_{n+2} & f_{n+1} & f_n \\ f_{n+2} & f_{n+3} & f_n & f_{n+1} \\ f_{n+1} & f_n & f_{n+3} & f_{n+2} \\ f_n & f_{n+1} & f_{n+2} & f_{n+3} \end{bmatrix} = f_{2n+1}f_{2n+7}.$$

However, once an appropriate acyclic directed graph is presented and understood, weights can be added to the arcs to generalize the identity as

$$\det \begin{bmatrix} G_{n+3} & G_{n+2} & G_{n+1} & G_n \\ G_{n+2} & G_{n+3} & G_n & G_{n+1} \\ H_{n+1} & H_n & H_{n+3} & H_{n+2} \\ H_n & H_{n+1} & H_{n+2} & H_{n+3} \end{bmatrix} = \begin{array}{l} (G_{n+2}H_{n+3} - G_nH_{n+1})\cdot \\ (H_{n+2}G_{n+3} - H_nG_{n+1}) - \\ 4(G_0G_2 - G_1^2)(H_0H_2 - H_1^2) \end{array}$$

where $\{G_n\}$ and $\{H_n\}$ are generalized Fibonacci sequences with arbitrary initial conditions.

The final work on display is a simple weighted digraph to instantly visualize Vandermonde's classic determinant,

$$\det \begin{bmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ 1 & x_2 & x_2^2 & \cdots & x_2^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{bmatrix} = \prod_{0 \le i \le j \le n} (x_j - x_i)$$

through nonintersecting lattice paths.

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