A New Exploration of the Lebesgue Identity

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1 Extended Abstract

In 1840 V. A. Lebesgue introduced what is now known as the Lebesgue identity [5],

$$\sum_{n=0}^{\infty} \frac{(-aq;q)_n}{(q;q)_n} q^{n(n+1)/2} = (-q;q)_\infty(-aq^2;q^2)_\infty \tag{1}$$

where we use the standard q-series notation,

$$(a;q)_n = \prod_{i=1}^{n}(1-aq^{i-1}), \tag{2}$$

$$(a;q)_\infty = \prod_{i=1}^{\infty}(1-aq^{i-1}). \tag{3}$$

With the use of partitions, we introduce a new combinatorial proof of the Lebesgue identity. It is with this new interpretation that we are able to find a new finite version of the identity,

$$\sum_{n=0}^{L} \begin{bmatrix} L \\ n \end{bmatrix}_q (-aq;q)_n q^{n(n+1)/2} = \sum_{k=0}^{L} \begin{bmatrix} L \\ k \end{bmatrix}_{q^2} (-q;q)_{L-k} a^k q^{k(k+1)}. \tag{4}$$

Since its introduction, the Lebesgue identity has been used to prove many partition identities by considering its special cases and interpreting the generating functions as counting sets of partitions. For the purpose of this talk we are mainly concerned with Sylvester's identity and the "little" Göllnitz theorems,

Theorem 1.1 (Sylvester)**.** Let $A_k(n)$ denote the number of partitions of n into odd parts (repititions allowed) such that exactly k different parts occur. Let $B_k(n)$ denote the number of partitions $\lambda = (\lambda_1, \cdots, \lambda_r)$ of n such that the sequence $(\lambda_1, \cdots, \lambda_r)$ is composed of exactly k noncontiguous sequences of one or more integers. Then $A_k(n) = B_k(n)$ for all k and n.

Theorem 1.2 (Göllnitz)**.** For each positive integer n, the partitions of n into parts congruent to $1, 5$ or $6 \mod 8$ are equinumerous with partitions of n with minimal difference two, and minimal difference four between odd parts.

Theorem 1.3 (Göllnitz)**.** For each positive integer n, the partitions of n into parts congruent to $2, 3$ or $7 \mod 8$ are equinumerous with partitions of n with minimal difference two, minimal difference four between odd parts, and the smallest part is greater than or equal to two.

In 1884 Sylvester's identity appeared in [7], but was not proven with the use of Lebesgue until 1972 in [6] by V. Ramamani and K. Ventkatachaliengar. Göllnitz's theorems appeared in [4] along with the more popular identities now known as the Gordon-Göllnitz identities. Time permitting, we introduce similar partition identities to those above by implementing the use of our finite version of Lebesgue.

It should be noted that partitions have been used in previous works to prove the Lebesgue identity. One such proof can be found in [2], and is considered indirect because of its use of a known map (Euler's theorem) in the first step of the bijection. A more direct proof can be found in [1] and is a modified version of the proof presented in [3].

References

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