Lattice paths and the Riordan group

Abstract

The Riordan group is a group of lower triangular matrices given by two generating functions, $g(z) = g_0 + g_1 z + g_2 z^2 + \cdots$ and $f(z) = f_1 z + f_2 z^2 + f_3 z^3 + \cdots$ where $f_1 g_0 \neq 0$. The $(n,k)_{n,k \geq 0}$ entry of the matrix is $[z^n]\, g(z)\,(f(z))^k$. The notation for this matrix is merely (g, f). Examples include the identity matrix $(1, z)$, the Pascal matrix $P = \left(\frac{1}{1-z}, \frac{z}{1-z}\right)$, and several matrices involving Catalan numbers such as (C, zC), $(1, zC^2)$, and $\left(C\left(z^2\right), zC\left(z^2\right)\right)$. Here $C = C(z) = \frac{1-\sqrt{1-4z}}{2z} = \sum \frac{1}{n+1}\binom{2n}{n} z^n$ is the generating function for the Catalan numbers. There are many more examples and all of them can be thought of in terms of lattice paths with appropriate weights.

There are many recent results involving the group structure. For instance it has been shown that all elements of order two are in the same conjugacy class.

An example that illlustrates a connection between lattice paths and the Riordan group starts with the following random walk problem. How many random walks are there using as steps $E = (1,0)$, $N = (0,1)$ and, only on the line $y = x$, $D = (1,1)$? The last condition is that the line never go above the line $y = x$. Let $G(z) = G$ be the generating function for such paths going from $(0,0)$ to (n,n). If we decompose the path by returns to the $y = x$ line we get the matrix

$$(1, z(1+C)) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \\ 0 & 2 & 0 & 0 & 0 & 0 & \\ 0 & 1 & 4 & 0 & 0 & 0 & \\ 0 & 2 & 4 & 8 & 0 & 0 & \cdots \\ 0 & 5 & 9 & 12 & 16 & 0 & \\ 0 & 14 & 24 & 30 & 32 & 32 & \\ & & & \cdots & & & \end{bmatrix}$$

One of the important subgroups of the Riordan group is the associated subgroup, all matrices of the form $(1, f)$.It turns out that this matrix is the only element in the associated subgroup with row sums $(C_{n+1})_{n \geq 0}$ and also it turns out that $G = C^2$. If we allow the path to go above the line $y = x$ the generating function becomes $\frac{1}{1-z(1+2C)} = 1 + 3z + 11z^2 + 43z^3 + 173z^4 + \cdots$. This is known in the recreational literature as the Manhattan taxicab problem including Broadway as the diagonal street. See Sloane's EIS A026670 for more references and alternate forms.

In this talk we will discuss this and other recent results linking random walks with the Riordan group and present open questions as well.