A Direct Computer Proof of Two Special Function Identities

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In his analytic work [4], E. Symeonidis derived in an indirect fashion two interesting special function identities involving Gegenbauer polynomials. In [2] the question of direct proofs was posed. We answer this question by presenting direct proofs obtained with the help of computer algebra algorithms based on WZ theory. In this extended abstract we give sketches of our proofs; the underlying strategy might be applicable to other special function identities of similar type.

1 Introduction

The following two special function identities were obtained in [4] by computing different expressions for the same Poisson kernel for a ball in non-euclidean spaces. For $|x| < 1$, $|t| < 1$ and $n \in \mathbb{N}$ with $n \geq 3$:

$$\sum_{k\geq 0} \frac{\binom{k+n-2}{k}}{\binom{k+\frac{n}{2}-2}{k}} t^k \, {}_2F_1\left(k, 1-\frac{n}{2}; k+\frac{n}{2}; t^2\right) C_k^{\frac{n-2}{2}}(x) = \left(\tfrac{1-t^2}{1-2tx+t^2}\right)^{n-1}, \tag{1}$$

$$\sum_{k\geq 0} \frac{\Gamma\left(\frac{k+n}{2}\right)\Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{n}{2}+k-1\right)} t^k \, {}_2F_1\left(k, k+n-1; k+\frac{n}{2}; \tfrac{1-\sqrt{1-t^2}}{2}\right) C_k^{\frac{n-2}{2}}(x) =$$

$$= \frac{(n-2)\Gamma\left(\frac{n+1}{2}\right)}{2\Gamma\left(\frac{n}{2}+1\right)} \sqrt{1-t^2} \, {}_2F_1\left(n, 1; \frac{n}{2}+1; \tfrac{xt+1}{2}\right), \tag{2}$$

where

$$C_k^{\lambda}(x) = \frac{(\lambda)_k}{\Gamma(k+1)} (2x)^k \, {}_2F_1\left(-\frac{k}{2}, \frac{1-k}{2}; 1-k-\lambda; \frac{1}{x^2}\right) \tag{3}$$

denotes the ultraspherical or Gegenbauer polynomials.

The basic idea for our proofs, is to find a recurrence that is satisfied by both sides of the identity and then check the equality of finitely many initial values.

The two sides are multiple sums of the form $\sum_{k_1} \cdots \sum_{k_r} \mathcal{F}(n, k_1, \ldots, k_r, \alpha)$ with summand $\mathcal{F}(n, k, \alpha)$ which is hypergeometric in the variables $n = (n_1, \ldots, n_l)$ and $k = (k_1, \ldots, k_r)$ and has additional parameters $\alpha = (\alpha_1, \ldots, \alpha_s)$. Recurrences for such sums are found using the Mathematica implementation of Wegschaider's algorithm [5] which is an extension of multivariate WZ summation [6].

In[1]:= **<< MultiSum.m**

MultiSum Package by Kurt Wegschaider (enhanced by Axel Riese and Burkhard Zimmermann) – © RISC Linz – V2.02β (02/21/05)

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However, it is advisable to keep the size of the input of the algorithm as small as possible, otherwise very large linear equation systems over a field of rational functions need to be solved. To this purpose, coefficient comparison with respect to components of α are a strategy to eliminate summation quantifiers and to reduce the number of variables such that the identities that need to be proven become smaller. In most cases, we can reduce the problem to a single summation problem and use the more efficient Zeilberger's algorithm [3]. But, straight-forward coefficient comparison does not always work and some change of variables is necessary.

2 Proof of the First Identity

We observe that a change of variable $1-x =: y$ is useful when dealing with:

$$\left(1-2tx+t^2\right)^{n-1} = \left((1-t)^2+2ty\right)^{n-1} = \sum_{l=0}^{n-1}\binom{n-1}{l}(2ty)^{n-1-l}(1-t)^{2l}$$

while for the Gegenbauer polynomials we use the following representation:

$$C_k^{\frac{n-2}{2}}(x) = \binom{k+n-3}{k}\,{}_2F_1\left(-k,k+n-2;\frac{n-1}{2};\frac{1-x}{2}\right)$$

Then (1) becomes:

$$\sum_{k\geq 0}\frac{\binom{k+n-2}{k}}{\binom{k+\frac{n}{2}-2}{k}}t^k\sum_{j\geq 0}\frac{(k)_j\left(1-\frac{n}{2}\right)_j}{\left(k+\frac{n}{2}\right)_j\Gamma(j+1)}t^{2j}\binom{k+n-3}{k}\sum_{i\geq 0}\frac{(-k)_i\,(k+n-2)_i}{\left(\frac{n-1}{2}\right)_i\Gamma(i+1)2^i}y^i$$
$$\times\sum_{l=0}^{n-1}\binom{n-1}{l}(2ty)^{n-1-l}(1-t)^{2l} = \left(1-t^2\right)^{n-1} \tag{4}$$

for $y\geq 0$, $|t|<1$ and $n\in\mathbb{N}$ with $n\geq 3$.
Coefficient comparison with respect to y leads to the following case distinction:

(a) The constant coefficient is obtained when $l=n-1$ and $i=0$ and it remains to prove:

$$\sum_{k\geq 0}\frac{\binom{k+n-2}{k}\binom{k+n-3}{k}}{\binom{k+\frac{n}{2}-2}{k}}t^k\sum_{j\geq 0}\frac{(k)_j\left(1-\frac{n}{2}\right)_j}{\left(k+\frac{n}{2}\right)_j\Gamma(j+1)}t^{2j} = \left(\frac{1+t}{1-t}\right)^{n-1} \tag{5}$$

The right hand side of (5) is:

$$\left(\frac{1+t}{1-t}\right)^{n-1} = (-1)^{n-1}\left(1-\frac{2}{1-t}\right)^{n-1} = (-1)^{n-1}\sum_{m\geq 0}(-t)^m\sum_{s=0}^{n-1}\binom{-s}{m}\binom{n-1}{s}(-2)^s$$

and via coefficient comparison with respect to t, for any $m\geq 0$, (5) becomes:

$$\sum_{j\geq 0}\frac{\binom{k+n-2}{k}\binom{k+n-3}{k}}{\binom{k+\frac{n}{2}-2}{k}}\frac{(k)_j\left(1-\frac{n}{2}\right)_j}{\left(k+\frac{n}{2}\right)_j\Gamma(j+1)} \overset{k=m-2j}{=} (-1)^{n-1+m}\sum_{s=0}^{n-1}\binom{-s}{m}\binom{n-1}{s}(-2)^s \tag{6}$$

Using Zeilberger's algorithm, we get for both sides of the identity the same recurrence:

$$m\mathrm{SUM}[m] + 2(n-1)\mathrm{SUM}[1+m] - (2+m)\mathrm{SUM}[2+m] = 0$$

and it is trivial to check that (6) holds for $m = 0$ and $m = 1$.

(b) If $f \geq 1$ is an arbitrary power of y then $l = n - 1 + i - f$ and we need to prove the following:

$$\sum_{k\geq 0} \frac{\binom{k+n-2}{k}\binom{k+n-3}{k}}{\binom{k+\frac{n}{2}-2}{k}} t^k \sum_{j\geq 0} \frac{(k)_j \left(1-\frac{n}{2}\right)_j}{\left(k+\frac{n}{2}\right)_j \Gamma(j+1)} t^{2j} \sum_{i=0}^{f} \frac{(-k)_i (k+n-2)_i}{\left(\frac{n-1}{2}\right)_i \Gamma(i+1) 4^i} \binom{n-1}{f-i} t^{f-i}(1-t)^{2i} = 0 \quad (7)$$

Moreover, an arbitrary t^p, $p \geq 0$ has the coefficient :

$$\sum_{k\geq 0} \frac{\binom{k+n-2}{k}\binom{k+n-3}{k}}{\binom{k+\frac{n}{2}-2}{k}} \sum_{j\geq 0} \frac{(k)_j \left(1-\frac{n}{2}\right)_j}{\left(k+\frac{n}{2}\right)_j \Gamma(j+1)} \sum_{i=0}^{f} \frac{(-k)_i (k+n-2)_i}{\left(\frac{n-1}{2}\right)_i \Gamma(i+1) 4^i} \binom{n-1}{f-i} \binom{2i}{m} (-1)^m \underset{m=p-f+i-k-2j}{=} 0$$

A recurrence in the parameters $f \geq 1$ and $p \geq 0$ corresponds to proving (4) by induction with respect to arbitrary powers of y and of t. Wegschaider's algorithm gives us such a recurrence:

$$\begin{aligned}
&(f-p)(3f-n-p)\mathrm{SUM}[f,p] + (5f^2 - 9nf + 10f + 2n^2 - p^2 - n + 3np - 6p - 3)\mathrm{SUM}[f,p+1]+ \\
&+ (f^2 - 3nf + 4pf + 10f + 2n^2 - p^2 - 10n - 3np + 6)\mathrm{SUM}[f,p+2] - (f-p-3)(f-n+p+3) \\
&\mathrm{SUM}[f,p+3] + (f-p)(3f+n-p)\mathrm{SUM}[f+1,p+1] + (9f^2 - 5nf + 20f - 2n^2 - p^2 + n+ \\
&+ np - 6p + 1)\mathrm{SUM}[f+1,p+2] + (5f^2 - 3nf + 4pf + 20f - 2n^2 - p^2 - 2n - np + 10) \\
&\mathrm{SUM}[f+1,p+3] - (f-p-3)(f+n+p+3)\mathrm{SUM}[f+1,p+4] + 2(f+2)(2f+n+1) \\
&\mathrm{SUM}[f+2,p+3] + 2(f+2)(2f+n+1)\mathrm{SUM}[f+2,p+4] = 0
\end{aligned}$$

It is trivial to check the necessary initial values: $\mathrm{SUM}[1,p]$, $\mathrm{SUM}[2,p]$, for all $p \geq 0$ and $\mathrm{SUM}[f,0]$, $\mathrm{SUM}[f,2]$, $\mathrm{SUM}[f,2]$, $\mathrm{SUM}[f,3]$, for all $f \geq 1$.

3 Proof of the Second Identity

Using (3) and after some simplifications we only need to prove that

$$\sum_{k\geq 0} \frac{\Gamma\left(\frac{k+n}{2}\right)\Gamma\left(\frac{k+1}{2}\right)}{\Gamma(k+1)} (2xt)^k \sum_{j\geq 0} \frac{(k)_j (k+n-1)_j}{\left(\frac{n}{2}+k\right)_j \Gamma(j+1) 2^j} \left(1-\sqrt{1-t^2}\right)^j$$
$$\sum_{i\geq 0} \frac{\left(-\frac{k}{2}\right)_i \left(\frac{1-k}{2}\right)_i}{\left(2-\frac{n}{2}-k\right)_i \Gamma(i+1)} x^{-2i} = \frac{2}{n}\Gamma\left(\frac{n+1}{2}\right)\sqrt{1-t^2} \sum_{s\geq 0} \frac{(n)_s}{\left(\frac{n}{2}+1\right)_s 2^s} (1+xt)^s$$

holds for all $|x| < 1$, $|t| < 1$ and $n \in \mathbb{N}$ with $n \geq 3$.
Coefficient comparison for arbitrary x^m with $m \geq 0$ leads to:

$$\sum_{i\geq 0} \frac{\Gamma\left(\frac{k+n}{2}\right)\Gamma\left(\frac{k+1}{2}\right)}{\Gamma(k+1)} (2xt)^k \frac{\left(-\frac{k}{2}\right)_i \left(\frac{1-k}{2}\right)_i}{\left(2-\frac{n}{2}-k\right)_i \Gamma(i+1)} \sum_{j\geq 0} \frac{(k)_j (k+n-1)_j}{\left(\frac{n}{2}+k\right)_j \Gamma(j+1) 2^j} \left(1-\sqrt{1-t^2}\right)^j =$$
$$\overset{k=m+2i}{=} \frac{2}{n}\Gamma\left(\frac{n+1}{2}\right) t^m \sqrt{1-t^2} \sum_{s\geq m} \binom{s}{m} \frac{(n)_s}{\left(\frac{n}{2}+1\right)_s 2^s} \quad (8)$$

Treating the case $t = 0$ separately, we denote $\alpha := \sqrt{1-t^2}$ (then $t^2 = 1 - \alpha^2$) and we prepare the stage for the coefficient comparison with respect to α. So (8) becomes:

$$\sum_{r\geq 0} (-1)^r \alpha^{2r} \sum_{i\geq r} \binom{i}{r} \frac{\Gamma\left(\frac{k+n}{2}\right)\Gamma\left(\frac{k+1}{2}\right)}{\Gamma(k+1)} 2^k \frac{\left(-\frac{k}{2}\right)_i \left(\frac{1-k}{2}\right)_i}{\left(2-\frac{n}{2}-k\right)_i \Gamma(i+1)} \sum_{l\geq 0} (-\alpha)^l$$
$$\times \sum_{j\geq l} \binom{j}{l} \frac{(k)_j (k+n-1)_j}{\left(\frac{n}{2}+k\right)_j \Gamma(j+1) 2^j} \overset{k=m+2i}{=} \frac{2}{n}\Gamma\left(\frac{n+1}{2}\right) \alpha \sum_{s\geq m} \binom{s}{m} \frac{(n)_s}{\left(\frac{n}{2}+1\right)_s 2^s} \quad (9)$$

We first get a closed form for the most-inner sum of the left hand side, for arbitrary $l \geq 0$:

$$\sum_{j\geq l} \binom{j}{l} \frac{(k)_j (k+n-1)_j}{\left(\frac{n}{2}+k\right)_j \Gamma(j+1) 2^j} = \frac{(k)_l (k+n-1)_l}{\Gamma(l+1)2} \sqrt{\pi} \frac{\Gamma\left(k+\frac{n}{2}\right)}{\Gamma\left(\frac{k+l+1}{2}\right)\Gamma\left(\frac{k+l+n}{2}\right)}$$

then we proceed with the coefficient comparison with respect to α, analogous to the first identity.

4 Conclusion

Our proofs provide a verification of the work done in [4] and they are, to our knowledge, the first direct proofs for these identities. It is not clear how the strategy that we used to reduce the identities can be made algorithmic, since the need for changes in variables makes the problem hard. We suspect that coefficient comparisons with respect to free parameters are a good method for dealing with large special function identities like the ones presented.

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