Pattern Avoiding Dyck Paths, and Finite Operator Calculus

Part II: Patterns of General Length

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As in Part I of this series, we denote by $Dyck\,(n,m;p)$ the number of Dyck paths to (n,m) that avoid the pattern p, where $p\in\{u,d\}^r$. We can find $Dyck\,(n,m;p)$ for certain patterns for general $r\in\mathbb{N}$; we will present u^r, d^r, du^r, and others. The Finite Operator Calculus can ultimately expand a polynomial sequence only in terms of a "known" basic sequence; in many of the cases we find the numbers $Dyck\,(n,m;p)$ expanded in terms of "Eulerian coefficients" $\binom{x}{n}_r$, i.e., the coefficient of t^n in $(1+t+\cdots+t^{r-1})^x$. We will discuss some properties of these basic but interesting numbers.

In all cases we begin by converting the Dyck path to a ballot path $D\,(n,m)$ with step vectors North and East by "rotating" the Dyck path 45°. We have $D\,(n,m)=Dyck\,((m+n,m-n;p)$, omitting the chosen pattern from the notation. Note that $D\,(n,m)=0$ for all $m<n$. Of course, N substitutes for u, and E for d in the avoided pattern p for the ballot path. The most elementary example is given by $p=E^r$. Avoiding r East steps has no effect on the first $r-1$ columns, $D\,(0,m)\,,\ldots,D\,(0,r-1)$. If $n\geq r$, we actually avoid the pattern NE^r, because all previous columns avoided E^r. Hence $D\,(n,m)-D\,(n,m-1)=D\,(n-1,m)-D\,(n-r,m-1)$. This system of difference equations in m has a solution in the vector space of polynomials, because the extension of $D\,(0,m)$ to the polynomial $d_0\,(x)\equiv 1$ shows by induction over n that $D\,(n,m)=d_n(m)$, where $d_n(x)$ is a polynomial of degree n such that $d_n\,(x)-d_n\,(x-1)=d_{n-1}\,(x)-d_{n-r}\,(x-1)$.

m	1	8	35	110	270	536	866
7	1	7	27	75	161	273	357
6	1	6	20	48	87	118	104
5	1	5	14	28	40	36	0
4	1	4	9	14	13	0	-27
3	1	3	5	5	0	-10	-22
2	1	2	2	0	-4	-8	-10
1	1	1	0	-2	-3	-3	-2
0	1	0	-1	-2	0	0	0
	0	1	2	3	4	5	n

$D\,(n,m)$ (above the diagonal), and the polynomial extension $d_n\,(m)$

It is easy to verify that the recursion for $d_n\,(x)$ is also satisfied by the Euler coefficients $\binom{x}{n}_r$, but with the initial values $\binom{0}{n}_r=\delta_{0,n}$ instead of $d_n\,(n-1)=\delta_{0,n}$. The polynomials $d_n\,(x)$ are expanded in terms of $\binom{x}{n}_r$, giving

$$D\,(n,m)=d_n\,(m)=\frac{m+1-n}{m+1}\binom{m+1}{n}_r$$

Note that the Catalan numbers occur at position (n, n) for $n < r$, i.e.

$$C_n = \frac{1}{n+1} \binom{n+1}{n}_r$$

for all $0 \leq n < r$.