

# Products Imply Sums

Within almost all calculus classes, the addition rule for derivatives is generally presented before the multiplication rule. The addition rule is proved using the fact that the limit of the sum of two functions equals the sum of their respective limits, while the product rule requires us to assume this fact, together with the product rule for limits and the fact that a function differentiable at  $a$  is continuous there. The multiplication rule is thus true under more stringent hypotheses. Could it be the case that the multiplication rule actually implies the addition rule? While in the throes of an undergraduate research project on extensions of the Erdős-Ko-Rado theorem, we stumbled upon the proof of a special case of the above question: If the product rule and the chain rule (and thus the quotient rule) are true then any two functions  $f$  and  $g$  satisfying the benign conditions

- $(1 + (g/f))' = (g/f)'$ ;
- $f(x) \neq 0$  for any  $x$ ;

also satisfy the fact that  $(f + g)' = f' + g'$ . This result was new to us and several colleagues, and a proof follows:

Since  $f(x) \neq 0$  we may write

$$\phi(x) = f(x) + g(x) = f(x) \left( 1 + \frac{g(x)}{f(x)} \right). \quad (1)$$

The product rule, together with the assumption that  $(1 + (g/f))' = (g/f)'$ , yields the fact that

$$\phi'(x) = f(x) \left( 1 + \frac{g(x)}{f(x)} \right)' + f'(x) \left( 1 + \frac{g(x)}{f(x)} \right) = f(x) \left( \frac{g(x)}{f(x)} \right)' + f'(x) \left( 1 + \frac{g(x)}{f(x)} \right).$$

Finally using the quotient rule we see that

$$\begin{aligned} \phi'(x) &= f(x) \left( \frac{f(x)g'(x) - g(x)f'(x)}{f^2(x)} \right) + f'(x) \left( 1 + \frac{g(x)}{f(x)} \right) \\ &= \frac{f^2(x)g'(x) - g(x)f'(x)f(x)}{f^2(x)} + f'(x) + \frac{f'(x)g(x)}{f(x)} \\ &= g'(x) + f'(x), \end{aligned}$$

as asserted. The second author plans on weaving this nugget into his next Calculus I and Real Analysis classes.