Kurtosis orderings for two-sided power distributions

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Abstract

Kurtosis orderings for symmetric two-sided power (TSP) distributions are studied using the Van Zwet criterion. TSP symmetric distributions are kurtosis ordered and the relationship in terms of kurtosis between two TSP symmetric distributions depends on the ratio of the values of the shape parameter. Symmetric TSP distributions are found to have smaller kurtosis than the Laplace distributions and to be pairwise ordered with symmetric Beta distributions. The kurtosis of TSP distributions sharing the same parameter is explored by means of spread-spread plots. The results obtained shed some light on the kurtosises of Beta, Laplace, Triangular and Power distributions.

Key Words: Beta distribution, Laplace distribution, triangular distribution, Spread-Spread plots, Van Zwet criterion.

1. Introduction

Kurtosis is a delicate property of distributions related to their center and tails that has an impact on the performance of inference procedures for the variance. The first three moments of a distribution determine its location, spread and asymmetry, respectively. This turns out to be usually insufficient to fully characterize a distribution. Theoretically, all the existing moments are required for characterization (under certain mild assumptions.) The fourth moment is included in the kurtosis measure (β_2) suggested by K. Pearson (1905). Several other measures to account for the 'peakedness' and 'tails' of a distribution have been proposed more recently; all of them are designated as kurtosis related measures, e.g. Hosking (1990), Balanda and MacGillivray (1990), Groenveld (1998), Seier and Bonett (2003); and others. A unique way of assessing kurtosis is not available. It is therefore appropriate to determine kurtosis orderings between two distributions via their c.d.fs. Obviously the kurtosis ordering of a family of distributions, or between two related families, provides information about the structure of distributions in the family. A knowledge of the kurtosis of a distribution can also shed light on the performance of some tests of hypotheses for variances, or the power of normality tests when the observations are generated by that distribution.

The Two-sided (two parameter) Power (TSP) distribution was defined originally in Van Dorp and Kotz (2002) and studied in Kotz and Van Dorp (2005). It was originally introduced as a non-smooth alternative to the Beta distribution and found applications to finance, engineering and operation research problems (e.g. PERT). In its standard form, the density $f(.|\theta, n)$ is given (for $0 \leq x \leq 1$) by

$$f(x|\theta, n) = \begin{cases} n(\frac{\mathrm{x}}{\theta})^{n-1} & for \quad 0 \leq x \leq \theta \\ n(\frac{1-\mathrm{x}}{1-\theta})^{n-1} & for \quad \theta \leq x \leq 1 \end{cases}$$

Here parameter θ is $0 \leq \theta \leq 1$ and $n > 0$ not necessarily an integer. The cdf is given by

$$F(x|\theta, n) = \begin{cases} \theta(\frac{\mathrm{x}}{\theta})^{n} & for \quad 0 \leq x \leq \theta \\ 1 - (1-\theta)(\frac{1-\mathrm{x}}{1-\theta})^{n} & for \quad \theta \leq x \leq 1 \end{cases}$$

Evidently $F^{-1}(y) = \sqrt[n]{y\theta^{n-1}}$ for $0 \le y \le \theta$ and $F^{-1}(y) = 1 - \sqrt[n]{(1-y)(1-\theta)^{n-1}}$ for $\theta \le y \le 1$. A four parameter version is defined on $a \le x \le b$, $b \ge a$. For $n > 2$, the symmetric TSP resembles the central part of the Laplace distribution. In Kotz and Van Dorp (2005) the kurtosis of TSP distributions was studied using kurtosis measure β_2. However, as mentioned above, distributions can also be compared in terms of kurtosis using their c.d.fs. Indeed, Van Zwet (1964) proposed a criterion to compare symmetric distributions in terms of kurtosis and proved that the family of distributions $Beta(\alpha, \alpha)$ is ordered according to the parameter α. His criterion states that F has smaller kurtosis than G iff $h(x) = G^{-1}(F(x))$ is convex, to the right of the common point of symmetry. Any measure of kurtosis ought to satisfy kurtosis orderings as established by $G^{-1}(F(x))$. Not all symmetric distributions are kurtosis ordered in the Van Zwet sense but Groenveld (1998) provides a list of distributions that are ordered according to this criterion. The spread-spread plots developed by Balanda and McGillivray (1990) extended the feasibility of establishing kurtosis orderings to non-symmetric distributions. Here we shall establish kurtosis orderings among TSP symmetric distributions and between them and the Beta and Laplace distributions. The kurtosis of non-symmetric versions of the TSP distribution is explored by means of spread-spread plots.

2. Ordering of symmetric Two-sided Power distributions

Figure 1 displays the standard two-sided power distribution for $\theta = 0.5$ and a number of values of the parameter $n = .5(.5)3.5$. For $\theta = 0.5$, $F(x) = 2^{n-1}x^n$ for $0 \le x \le 0.5$ and $F(x) = 1 - 2^{n-1}(1-x)^n$ for $0.5 \le x \le 1$.

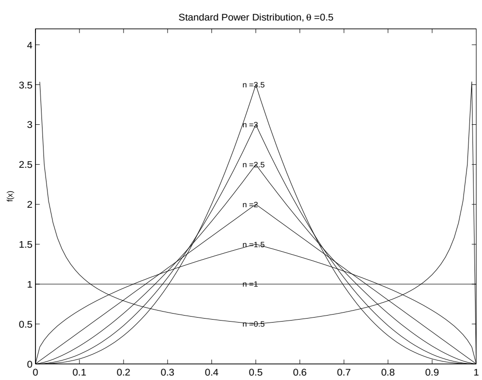

Figure 1. Pdfs for the symmetric standard $TSP(0.5, n)$ distributions

Consider two TSP distributions with parameters $\theta = 0.5$ and $n_1 < n_2$. The common point of symmetry is 0.5, F(0.5)=0.5 and $F^{-1}(y) = 1 - (0.5)\sqrt[n]{2(1-y)}$ for $0.5 \le y \le 1$. Thus $h(x) = G^{-1}(F(x)) = 1 - 2^{k-1}(1-x)^k$ for $k = n_1/n_2$ and $0.5 \le x \le 1$. The first derivative $h'(x) = 2^{k-1}k(1-x)^{k-1}$ is non-negative for $0 \le x \le 1$ as well as the second derivative $h''(x) = 2^{k-1}k(1-k)(1-x)^{k-2}$ when restricted for $0.5 \le x \le 1$. Thus $G_{n_2}^{-1}F_{n_1}(x)$ is convex for $n_1 < n_2$ and the TSP symmetric distributions are indeed kurtosis ordered according to the value of the parameter n. Hence, the relationship between the kurtosis of two symmetric TSP distributions seems to depend solely on the ratio $k = n_1/n_2$. Figure 2 displays the function $G_{n_1/k}^{-1}F_{n_1}(x)$, for a number of values of the ratio $k = n_1/n_2$ and $0.5 \le x \le 1$. As k increases from 0.1 to 1 the convexity of the function decreases.

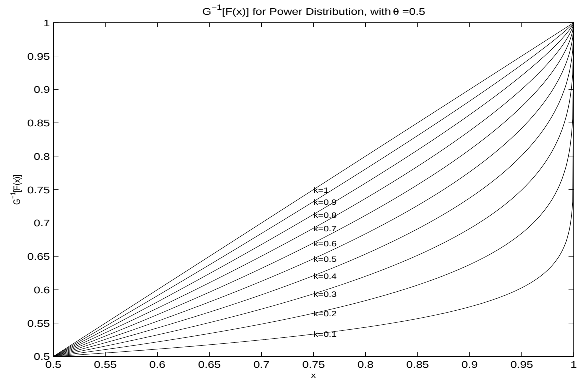

Figure 2. $G^{-1}_{n1/k}F_{n1}(x)$ for symmetric TSP distributions for $k = .1(.1)1$.

In the four parameter version of the $TSP(a, m, b, n)$, x takes values in the interval (a, b) and m is the mode (or antimode) of the density. For symmetric distributions with mode 0 ($m = 0$ and $a = -b$), the density function is

$$f(x|b, 0, b, n) = \begin{cases} \frac{n(x+b)^{n-1}}{2b^n} & for \quad -b \le x \le 0 \\ \frac{n(b-x)^{n-1}}{2b^n} & for \quad 0 \le x \le b \end{cases}$$

Figure 3 displays $f(x|-b, 0, b, n)$ for $n = 4$ and b=1(1)3. In this case, for two different values of the parameter b (b_1 and b_2), $h(x) = G^{-1}_{b_2}(F_{b_1}(x)) = (b_2/b_1)x$. Thus, two symmetric TSP distributions with parameter n and mode 0, one defined over $(-b, b)$ and the other defined over $(-1/2, 1/2)$, have the same kurtosis.

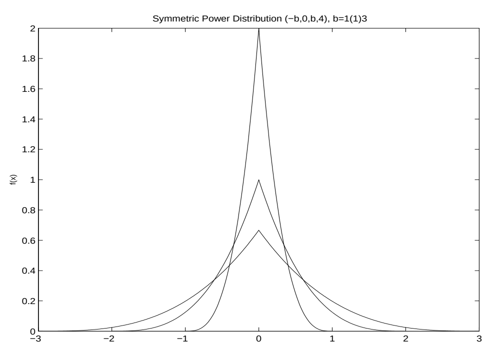

Figure 3. Symmetric TSP distributions with mode 0,$n = 4$ and $|x| \le b$

3. Comparing the kurtosis of symmetric TSP and Beta distributions

Let G be a $TSP(0.5, n)$ distribution and F be the $Beta(n, n)$ distribution. In this case $h = G^{-1}(F(x)) = 1 - 0.5\sqrt[n]{(2(1-y)}$ where $y = I(n, n) = \frac{1}{B(n,n)}\int_0^x t^{n-1}(1-t)^{n-1}dt$, an incomplete Beta function for $0.5 \leq x \leq 1$. The first derivative of y as a function of x is: $y' = \frac{1}{B(n,n)} x^{n-1}(1-x)^{n-1}$, hence $h' = \frac{2^{1/n}}{2nB(n,n)} x^{n-1}(1-x)^{n-1}(1-I(n,n))^{(1-n)/n}$. Also $h'' = \frac{(1-I(n,n))^{1/n-2} 2^{1/n} x^{n-2}(1-x)^{n-2}(n-1)}{2nB(n,n)} \{(1-2x)(1-I(n,n)) + \frac{1}{nB(n,n)} x^n (1-x)^n\}$. The sign of the second derivative depends on $n-1$. Note that $I(n,n)$ for $0 < n < 1$ is of totally different form than $I(n,n)$ for $n > 1$. Thus, $TSP(0.5, n)$ has a larger kurtosis than $Beta(n,n)$ for $n > 1$ and a smaller kurtosis for $0 < n < 1$, regardless of how the kurtosis is measured. This fact could be helpful when trying to decide between the two models to represent the behavior of a variable. From Figure 4 it is evident that as n departs from 1, the curve becomes more convex, suggesting that the inequality in kurtosis between the $TSP(0.5, n)$ and the $Beta(n,n)$ distributions increases.

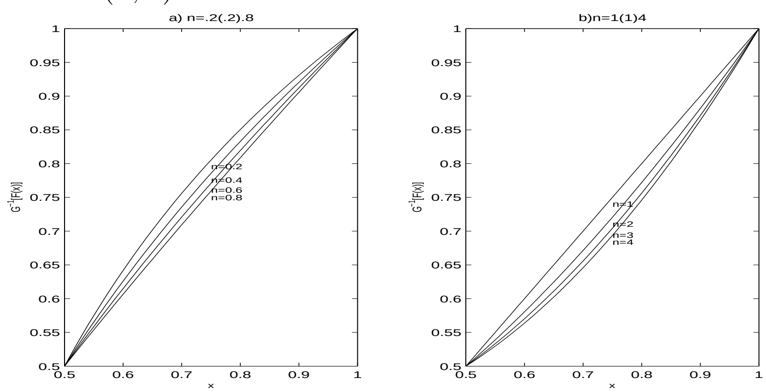

Figure 4. $G^{-1}F(x)$ for symmetric TSP and Beta distributions a)$n = .2(.2).8$. b)$n = 1(1)4$.

4. Comparing the TSP with the Laplace distribution in terms of kurtosis

The TSP distributions serve, when $n > 1$, as a peaked alternative to the Beta distribution analogously to the Laplace being a peaked alternative to the normal. For $n > 2$ the symmetric TSP and the Laplace distribution have a similar shape at the center of the distribution. However, the TSP distribution has a bounded domain while the Laplace density is defined over an unbounded domain. Consider a TSP distribution with mode 0 and the range $-0.5 \leq x \leq 0.5$. Here $y = F(x) = 1 - 0.5(1-2x)^n$ for $x \geq 0$. The standard Laplace distribution with density function $f(x) = (1/2)exp(-|x|)$, for $-\infty < x < \infty$, has inverse cdf $G^{-1}(y) = -ln(1-2(y-0.5))$ for $y > 0.5$. Thus $G^{-1}(F(x)) = -ln((1-2x)^n)$. For $0 \leq x < 0.5$ the first derivative of $h(x) = G^{-1}(F(x))$ is $2n/(1-2x)$ and the second is $4n/(1-2x)^2$. Both derivatives are positive, thus the four parameter $TSP(-1/2, 0, 1/2, n)$ distribution has smaller kurtosis than the Laplace distribution. The Laplace distribution has been extensively used in simulations to study the performance of inference tools. In Seier(2002) the empirical power of several normality tests was calculated by simulation. In particular, the empirical power (0.2,0.52 and 0.74 for sample sizes 20,50 and 100 respectively) of a skewness-kurtosis based test (D'Agostino et al, 1990) was reported. We would expect the test to be less powerful against a TSP alternative, regardless of the value of the parameter n, than against a Laplace alternative. In a different context when high kurtosis is a liability, we would expect non-robust tests for equality of variances to have a better performance when data are generated by a symmetric TSP distribution rather than by a Laplace distribution.

5. Non-symmetric standard TSP distributions with common θ

Two-sided power distributions with the same location parameter $\theta \neq 0.5$ are skewed distributions with mode θ . The d.fs of these distributions are a skewed version of the distributions presented in Figure 1; however, as above, n is the maximal value of the density in the standard case ($0 \leq x \leq 1$). Balanda and MacGillivray (1990) have defined the spread function $S_H(u) = H^{-1}(0.5+u)-H^{-1}(0.5-u)$, where H^{-1} is the inverse cdf of the distribution, for $0 \leq u \leq 1/2$. Two distributions are considered to be ordered with respect to kurtosis if the spread-spread plot S_{H_2} vs S_{H_1} results in a convex curve. The derivations below assume that $\theta < 0.5$, similar arguments can be used when $\theta > 0.5$. If $\theta < 0.5$ then $y = 0.5 + u > 0.5$, therefore $H^{-1}(0.5 + u)$ ought to be calculated using the expression $F^{-1}(y) = 1 - \sqrt[n]{(1-y)(1-\theta)^{n-1}}$ where $y = 0.5 + u$ so that $H^{-1}(0.5+u) = 1 - \sqrt[n]{(1-(0.5+u))(1-\theta)^{n-1}} = 1 - \sqrt[n]{(0.5-u))(1-\theta)^{n-1}}$. Since $0.5 - u < 0.5$ and $\theta < 0.5$, two cases $0.5 - u < \theta$ and $0.5 - u > \theta$ should be considered. For $u > 0.5 - \theta$ we have $H^{-1}(0.5-u) = \sqrt[n]{(0.5-u)\theta^{n-1}}$, while if $u < 0.5-\theta$, then $H^{-1}(0.5-u) = 1- \sqrt[n]{(0.5+u)(1-\theta)^{n-1}}$. Hence, for $\theta < 0.5$, the spread function is:
$S(u) = \sqrt[n]{(0.5+u)(1-\theta)^{n-1}} - \sqrt[n]{(0.5-u)(1-\theta)^{n-1}}$
$= (1-\theta)^{1-1/n}(\sqrt[n]{0.5+u} - \sqrt[n]{0.5-u}$ for $u \leq 0.5 - \theta$,
and $S(u) = 1 - \sqrt[n]{(0.5-u)(1-\theta)^{n-1}} - \sqrt[n]{(0.5-u)\theta^{n-1}}$
$= 1 - \sqrt[n]{0.5-u}((1-\theta)^{1-1/n} + \theta^{1-1/n})$ for $u \geq 0.5 - \theta$

Figure 5 displays the spread function for $\theta = 1/4$ and a number of values of the parameter $n = 1(.5)3.5$. For other values of the parameter $\theta \leq 0.5$, the spread function is of similar form but not identical.

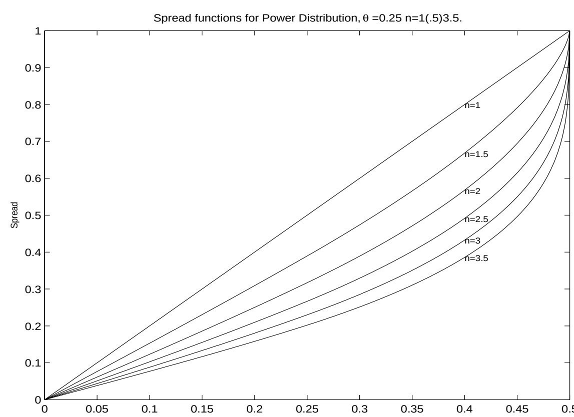

Figure 5. Spread functions of non-symmetric $TSP(0.25, n)$ distributions with $n = 1(.5)3.5$

For the special case when the mode is 0, the spread function evidently reduces to $S(u|\theta = 0) = \sqrt[n]{0.5+u} - \sqrt[n]{0.5-u}$ for $0 \leq u \leq 0.5$. Obviously, for the left triangular distribution with pdf $f(x) = 2(1-x)$, $S(u|n = 2, \theta = 0) = \sqrt{0.5+u} - \sqrt{0.5-u}$ for $0 \leq u \leq 0.5$. Figure 6 compares the distributions with $\theta = 0$ and several values of $n = 4(2)14$, with the left triangular distribution using the spread-spread plot. It is clear that standard TSP distributions with mode=0 and $n > 2$ have more kurtosis than the left triangular distribution; the relative increment in kurtosis for equal increments in n tends to decrease as n increases.

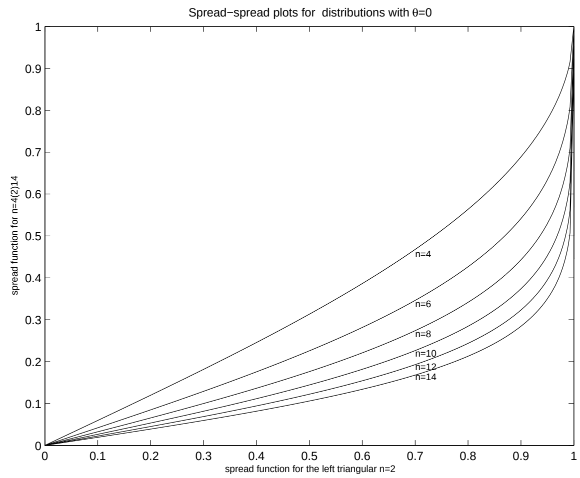

Figure 6. Comparison of $TSP(0, n)$ for $n = 4(2)14$ with a left triangular distribution

6. Conclusions

Van Zwet's criterion and spread-spread plots are useful tools to study the relationship among two-sided power (TSP) distributions as well as their relationship to other distributions.
For TSP symmetric distributions it was found that:

- they are ordered, via kurtosis, according to the value of the shape parameter n
- the relative kurtosis relationship of two symmetric TSP distributions depends on the ratio of their shape parameters n_1/n_2.
- changing the domain of definition but keeping n constant does not change the kurtosis.
- they are pairwise ordered with the $Beta(n, n)$ distributions: for $n > 1$ the TSP has larger kurtosis, and vice versa for $n < 1$. The difference in kurtosis between the symmetric TSP and the corresponding symmetric $Beta(n, n)$ seems to slightly increase as n departs from 1.
- they have smaller kurtosis than the Laplace distribution.Consequently, the power of normality tests and the performance of non-robust inference tools for the variance can be anticipated for the symmetric TSP distributions based on known results for the Laplace distribution. Normality tests are expected to be less powerful against a TSP alternative than against a Laplace alternative. Non-robust tests for equality of variances are expected to have a better performance when data are generated by a symmetric TSP distribution rather than by a Laplace distribution.

Non-symmetric $TSP(\theta, n)$ distributions with the same value of n and different value of θ would be expected to have similar (but not identical) kurtosis (except of course for n=1); the difference in kurtosis is associated to the value of n. In the same manner, the kurtosis relationship between two TSP distributions with common location parameter θ and different n depends on the values of both parameters.

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