Broadcasts in Graphs

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Abstract

We say that a function $f : V \to \{0, 1, \ldots, diam(G)\}$ is a *broadcast* if for every vertex $v \in V$, $f(v) \leq e(v)$, where $diam(G)$ denotes the diameter of G and $e(v)$ denotes the eccentricity of v. The *cost* of a broadcast is the value $f(V) = \Sigma_{v \in V} f(v)$. In this paper we introduce and study the minimum and maximum costs of several types of broadcasts in graphs, including dominating, independent and efficient broadcasts.

1 Introduction

In his 1958 book, Berge [2] introduced the *coefficient of external stability*, which was renamed the *domination number* by Ore in his 1962 book [12]. An application of domination was given by Liu in his 1968 book [9]. Liu discussed the concept of *dominance* in communication networks, where a dominating set represents a set of cities which, having broadcast stations, can broadcast messages to every city in the network. It was assumed, however, that a given broadcast station could only transmit messages to adjacent nodes.

Since the publication of these three books, nearly 2000 research papers have been published on domination in graphs. Over the past 40 years more than 80 domination related parameters have been defined, most of which are listed in the appendix of the book by Haynes, Hedetniemi and Slater [6]. But none of these models of domination have been based on the broadcast model of Liu, until recently when Erwin [3] defined a model in which broadcast stations have an associated cost (or transmission power, say in watts) which enables them to broadcast messages to nodes at distances greater than one.

In this paper we note the similarity to Liu's 1968 model and extend the study of broadcasts in graphs, which will be defined in Section 3.

It is worth noting that distance-k domination has been studied. In this model all vertices not in the dominating set must be within distance k of at least one vertex in the dominating set, for some fixed, nonnegative integer k. Thus, we can assume that every vertex v in a distance-k dominating set can broadcast messages to all vertices within distance k of v. Distance parameters are models for problems involving the placement of desirable objects (for example, radio stations, hospitals) within an acceptable distance of the population, or the placement of undesirable objects (for example, nuclear reactors, garbage dumps) at a maximum distance from the given population. Slater [13] introduced the general case where the "acceptable" distances can be different for each vertex in the dominating set. For a survey of results on distance domination in graphs the reader is referred to Henning (Chapter 12, [10]).

2 Terminology and Notation

Let $G = (V, E)$ be a graph of *order* $n = |V(G)|$ and *size* $m = |E(G)|$. For a vertex $v \in V$, the *open neighborhood* of v is the set $N(v) = \{u \in V \mid uv \in E\}$ and the *closed neighborhood* is the set $N[v] = N(v) \cup \{v\}$. For a set $S \subseteq V$, its *open neighborhood* is $N(S) = \cup_{v \in S} N(v)$ and its *closed neighborhood* is $N[S] = N(S) \cup S$.

A set $S \subseteq V$ is a *dominating set* if $N[S] = V$. The *domination number* $\gamma(G)$ and the *upper domination number* $\Gamma(G)$ are, respectively, the minimum and maximum cardinalities of a minimal dominating set. We call a dominating set of G of minimum cardinality a $\gamma(G)$*-set*, and one of maximum cardinality a $\Gamma(G)$*-set*. We use similar notation for other parameters, that is, for a generic parameter $\mu(G)$, we call a set satisfying the property for the parameter and having cardinality $\mu(G)$, a $\mu(G)$-set. A set S is *independent* if no two vertices in S are adjacent. The *independent domination number* $i(G)$ and the *vertex independence number* $\beta_0(G)$ are, respectively, the minimum and maximum cardinalities of a maximal independent set in G. A set S is a *packing* if for every vertex $v \in V$, $|N[v] \cap S| \leq 1$. The *packing number* $P(G)$ equals the maximum cardinality of a packing in G, while the *lower packing number* $p(G)$ equals the minimum cardinality of a maximal packing in G. The following well-known inequality chain [6] relates these invariants:

$$p(G) \leq P(G) \leq \gamma(G) \leq i(G) \leq \beta_0(G) \leq \Gamma(G). \tag{1}$$

Finally, a dominating set S is said to be *efficient* if for every vertex $v \in V$, $|N[v] \cap S| = 1$. Two basic facts about efficient dominating sets (see [1]) are well known in domination theory: (i) not every graph has an efficient dominating set, for example, the five-cycle does not have one, and (ii) if a graph G has an efficient dominating set, then every efficient dominating set in G has the same cardinality, which equals the domination number $\gamma(G)$. For a comprehensive study of domination and related invariants, see [6, 7].

In the next section we will show that the concept of broadcasting in graphs provides an immediate and natural generalization of the concepts of independence, domination and packing in graphs. Thus, it is possible to develop a general theory of broadcasting in graphs, which parallels the theory of domination in graphs. As we shall see, many new insights and general theorems exist in this more general theory that do not exist in the theory of domination. In particular we will explore the extent to which the inequality chain (1) has a counterpart for the broadcasting invariants which we will define.

3 Broadcasts in Graphs

A function $f : V \to \{0, 1, \ldots, diam(G)\}$ is a *broadcast* if for every vertex $v \in V$, $f(v) \leq e(v)$, where $diam(G)$ denotes the *diameter* of G and $e(v)$ denotes the *eccentricity* of vertex v. Since we want $diam(G) \geq 1$ to be finite, for the remainder of this paper, we use graph to mean a nontrivial connected graph.

Given a broadcast f, we define $N_f[u] = \{v \mid d(u,v) \leq f(u)\}$ to be the *broadcast neighborhood* of u. Further, we define $V^0 = \{v \mid f(v) = 0\}$ and $V^+ = V - V^0 = \{u \mid f(u) > 0\}$ (if there is some potential for ambiguity, then we shall let $V^+ = V^+_f$, $V^0 = V^0_f$, and so on). We say that every vertex in V^+ is a *broadcast vertex*, and the *broadcast neighborhood* $N_f[V^+] = \bigcup_{v \in V^+} N_f[v]$. If $u \in V^+$ is a broadcast vertex, $v \in V$ and $d(u,v) \leq f(u)$, then vertex v can *hear* a broadcast from vertex u. The set of vertices that a vertex $v \in V$ can hear is defined as $H(v) = \{u \in V^+ \mid d(u,v) \leq f(u)\}$. For a vertex $v \in V^+$, the *private f-neighborhood* $pn_f[v]$ is the set $\{u \in V \mid H(u) = \{v\}\}$. If $v \in pn_f[v]$, then we say that v is its own private f-neighbor. We define the *cost* of a broadcast to be $f(V) = \Sigma_{v \in V} f(v) = \Sigma_{v \in V^+} f(v)$.

A broadcast f of some type is said to be *minimal* (respectively, *maximal*) if there does not exist a broadcast $g \neq f$ of the same type such that for every $u \in V$, $g(u) \leq f(u)$ (respectively, $g(u) \geq f(u)$). Given two distinct broadcasts f and g, we say that $f \leq g$ (respectively, $f \geq g$) if and only if for every vertex $u \in V$, $f(u) \leq g(u)$ (respectively, $f(u) \geq g(u)$), for all $u \in V$.

Let $f_S : V \to \{0, 1\}$ be the characteristic function of a set $S \subseteq V$ of a graph G, that is, $f_S(u) = 1$ if $u \in S$, and $f_S(u) = 0$ otherwise. We will be interested in the characteristic functions of a variety of sets of vertices in a graph.

With this terminology we can define a number of different kinds of broadcasts including those introduced in [3].

3.1 Dominating broadcasts

As defined in [3], a broadcast f is *dominating* if $N_f[V^+] = V(G)$, or equivalently, if for every $v \in V$, $|H(v)| \geq 1$. The minimum cost $f(V)$ of a dominating broadcast f of a graph G is the *broadcast domination number* $\gamma_b(G)$. We say that a dominating broadcast f of cost equal to $\gamma_b(G)$ is a *γ_b-broadcast.* The *upper broadcast domination number* equals the maximum cost of a minimal dominating broadcast, and is denoted $\Gamma_b(G)$.

We now make two observations. First, the characteristic function f_S of any minimal dominating set S in a graph G is a minimal dominating broadcast. Second, let $u \in V$ be any vertex in a graph G, and let $f_u : V \to \{0, 1, 2, \ldots, diam(G)\}$ be defined by $f_u(u) = e(u)$; $f_u(v) = 0$, if $v \neq u$. Then the broadcast f_u is dominating, since every vertex can hear a broadcast from

u, and has cost $e(u)$. If u is a vertex in the *center* of G (that is, $e(u) = rad(G)$), we call the broadcast f_u a *radius broadcast*, while if u is in the *periphery* of G (that is, $e(u) = diam(G)$), then f_u is a *diameter broadcast*. Thus, we have:

Observation 1 *For any graph G,*

$$\gamma_b(G) \leq min\{\gamma(G), rad(G)\} \leq max\{\Gamma(G), diam(G)\} \leq \Gamma_b(G).$$

It was shown in [3] that for a path P_n, $\gamma_b(P_n) = \gamma(P_n) = \lceil n/3 \rceil$. Let $S(G)$ denote the subdivision graph of G, and consider $T = S(K_{1,t})$. Since no vertex of T dominates T, it follows that $\gamma_b(T) \geq 2$. And since a radius broadcast dominates T, it follows that $\gamma_b(T) \leq rad(T) = 2$. Thus, equality can be attained in $\gamma_b(G) \leq \min\{\gamma(G), rad(G)\}$. On the other hand, it was shown in [3] that the difference between $\min\{\gamma(G), rad(G)\}$ and $\gamma_b(G)$ can be arbitrarily large. An analogous result holds for the quantities $\max\{\Gamma(G), diam(G)\}$ and $\Gamma_b(G)$, as we now see. For a positive integer k, let H_k be the graph obtained by joining an endvertex of $S(K_{1,2+k})$, where $S(G)$ denotes the subdivision graph of G, to an endvertex of P_{2k}. Then $\Gamma(H_k) = 2k + 3$ and $diam(H_k) = 2k + 4$, so $\max\{\Gamma(H_k), diam(H_k)\} = 2k + 4$. Let v be that endvertex of P_{2k} that is not joined in H_k to a vertex of $S(K_{1,2+k})$. Then the broadcast $f : V(H_k) \to \{0, 1, 2k + 2\}$, defined as $f(v) = 2k + 2$, $f(x) = 1$ for every leaf x of H_k different from v and $f(x) = 0$ for every nonleaf vertex x of H_k is a minimal dominating broadcast on H_k with cost $3k + 3$. Hence, $\Gamma_b(H_k) \geq 3k + 3$, and we make the following observation.

Observation 2 *For every positive integer k,*

$$\Gamma_b(H_k) - max\{\Gamma(H_k), diam(H_k)\} \geq k - 1.$$

We now establish an upper bound on Γ_b that makes use of two previously established results. We say that a vertex or edge of G *lies between* two vertices u and v if that vertex or edge is on some u-v geodesic (shortest u-v path).

Theorem 3 [3] *Let f a dominating broadcast on a graph G. Then f is minimal if and only if the following two conditions are satisfied:*

(*i*) *for every vertex v with $f(v) \geq 2$, there exists a private f-neighbor of v that is at distance $f(v)$ from v, and,*

(*ii*) *for every vertex v with $f(v) = 1$, v has a private f-neighbor in $N[v]$.*

Lemma 4 [3] *Let f be a dominating broadcast on G, $u, v \in V^+$ with $u \neq v$, and let u_p, v_p be private f-neighbors of (respectively) u and v. For every pair x, y of vertices of G, if x lies between u and u_p and y lies between v and v_p, then $x \neq y$.*

Theorem 5 *If G is a graph of size m, then $\Gamma_b(G) \leq m$ with equality if and only if G is a nontrivial star or path.*

Proof. Let f be a Γ_b-broadcast of G. From Theorem 3, if $v \in V^+$, then v has a private f-neighbor (denoted v_p) such that either (i) $f(v) = d(v, v_p)$, or (ii) $f(v) = 1$ and $v = v_p$. Define a function ϵ on V^+ as follows: if $v \in V^+$ satisfies (i), then $\epsilon(v)$ is the set of all edges that lie between v and v_p (hence $|\epsilon(v)| \geq f(v)$), while if v satisfies (ii), then $\epsilon(v) = \{e_v\}$. We note that e_v exists and is incident to a vertex in V^0 because G is connected and v is its own private f-neighbor. Notice that $f(V) \leq \sum_{v \in V^+} |\epsilon(v)|$. In order to prove the result it thus suffices to show that for any pair u, v of distinct vertices of V^+, $\epsilon(u) \cap \epsilon(v) = \emptyset$. We now consider two cases:

Case 1: Both u and v satisfy (i). From Lemma 4, if x lies between u and u_p and y lies between v and v_p, then $x \neq y$. Consequently, no edge lies both between u and u_p and between v and v_p, so $\epsilon(u) \cap \epsilon(v) = \emptyset$.

Case 2: At least one of u and v, say v, satisfies (ii). Since v is its own private f-neighbor, the vertex v' that is joined to v by e_v has $f(v) = 0$, so it is not possible that $v' = u$ or that $e_v = e_x$ for some $x \neq v$. If v' lies on any u-u_p geodesic, then v is dominated by u, contradicting that v is its own private f-neighbor. Therefore, the vertex v' does not lie on any u-u_p geodesic, so e_v does not lie on any u-u_p geodesic. Thus, $\epsilon(u) \cap \epsilon(v) = \emptyset$.

Therefore, $\Gamma_b(G) = \sum_{v \in V^+} |\epsilon(v)| \leq m$.

To prove the characterization, first note that $\Gamma_b(P_n) \geq diam(P_n) = n - 1 = m$ and hence we have shown that $\Gamma_b(P_n) = m$. For a star $K_{1,m}$, we define the function f where each leaf u has $f(u) = 1$ and the center v has $f(v) = 0$. Then f is a minimal dominating broadcast, and so $\Gamma_b(K_{1,m}) \geq m$ and we have $\Gamma_b(K_{1,m}) = m$.

For the converse, assume that $\Gamma_b(G) = m$ for some graph G of size m. Then equality must hold throughout the counting process of the proof of the inequality, that is, $f(V) = \sum_{v \in V^+} |\epsilon(v) = m$. In particular, $|\epsilon(v)| = f(v)$ for all $v \in V^+$ and each edge of $E(G)$ is in a set $e(v)$ for some $v \in V^+$. Moreover, the proof of our inequality shows that $\{e(v) \mid v \in V^+\}$ is a partition of $E(G)$. Since every edge in $e(v)$ is incident to at least one vertex in V^0 for all $v \in V^+$, it follows that V^+ is an independent set.

Assume that $v \in V^+$ and v satisfies (i). Since $|e(v)| = f(v)$, there is a unique v-v_p geodesic. Suppose that x lies between v and v_p and x has a neighbor y not on the v-v_p geodesic. Then $xy \in e(u)$ for some $u \neq v$, that is x lies between u and u_p, contradicting Lemma 4. Hence the connectivity of G implies that $V^+ = \{v\}$ and G is the v-v_p path.

Therefore we may assume that every vertex in V^+ satisfies (ii), that is, $e(v) = \{e_v\}$ for each $v \in V^+$. But then the connectivity of G implies that G is a star. □

In subsequent sections, we will consider the relationships between $\gamma_b(G)$, $\Gamma_b(G)$ and other broadcasting invariants. We conclude this section with examples obtaining equality in $\Gamma_b \geq \max\{diam(G), \Gamma(G)\}$. From Observation 1 and Theorem 5, for the path P_n, we have $diam(P_n) = n - 1 \leq \Gamma_b(P_n) \leq m = n - 1$ and hence, $\Gamma_b(P_n) = n - 1$.

Next consider the Petersen graph PG, and let S be the set of vertices on one of the five cycles of PG. Then S is a $\Gamma(PG)$-set, and so $\Gamma_b(PG) \geq \Gamma(PG) = 5$. Let f be a Γ_b-broadcast of PG. If $f(u) \geq 2$ for any $v \in V^+$, then the minimality of f implies that $f(v) = 0$ for all $v \neq u$. Thus, $f(V) = 2$. Hence we may assume that $f(u) = 1$ for all $u \in V^+$ implying that $\Gamma_b(PG) = \Gamma(PG) = 5$.

3.2 Independent broadcasts

A broadcast f is called *independent* if for every vertex $v \in V^+$, $N_f[v] \cap V^+ = \{v\}$, or equivalently, $|H(v)| = 1$. If f is an independent broadcast, then no broadcast vertex can hear a broadcast from any other broadcast vertex. We note that an independent broadcast need not be a dominating broadcast. The maximum cost of an independent broadcast of G is called the *broadcast independence number* and is denoted $\beta_b(G)$. The *lower broadcast independence number* $i_b(G)$ equals the minimum cost of a maximal independent broadcast of G.

Observation 6 *For any graph G, $i_b(G) \leq rad(G) \leq diam(G) \leq \beta_b(G)$.*

Proof. A radius broadcast and a diameter broadcast are both maximal independent broadcasts. $\square$

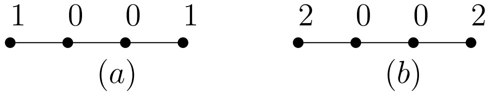
Let M be a subset of the peripheral vertices of G such that for any pair of vertices u and v in M, $d(u,v) = diam(G)$, and let $\mu(G)$ denote the maximum cardinality of such a subset of G. Our next observation improves the lower bound on $\beta_b(G)$.

Proposition 7 *For any graph G, $\beta_b(G) \geq \mu(G)(diam(G)-1) \geq 2(diam(G)-1)$ and this bound is sharp.*

Proof. For each vertex $v \in M$, let $f(v) = diam(G) - 1$; and for each vertex $u \in V - M$, let $f(u) = 0$. Then f is an independent broadcast implying that $\beta_b(G) \geq \mu(G)(diam(G)-1)$.

Next we show that the subdivided star $T = S(K_{1,t})$ with $t \geq 2$ attains the bound. First note that $diam(T) = 4$ and $\mu(T) = t$, so we have just shown that $\beta_b(T) \geq 3t$. To see that $\beta_b(T) \leq 3t$, let f be a β_b-broadcast of T. If $f(v) = 4$ for some $v \in V^+$, then $V^+ = \{v\}$ and $f(V) = 4 < 3t$. Thus assume that $f(v) \leq 3$ for all $v \in V^+$. Let x be the center of T. Note that $f(x) \leq e(x) = 2$. If $f(x) = 2$, then $f(v) = 0$ for all $v \in V - \{x\}$. If $f(x) = 1$, then $f(u) = 0$ for all $u \in N(x)$ and $f(v) \leq 1$ for all $v \in V - N[x]$. In either case, $f(V) < 3t$. Hence, $f(x) = 0$. Since at most one of a leaf and its support is in V^+, it follows that $|V^+| \leq t$ and hence, $f(V) \leq 3t$. $\square$

Note that the characteristic function f_S of any maximal independent set S in a graph G is an independent broadcast, and therefore $i(G) \leq \beta_0(G) \leq \beta_b(G)$, but f_S is not necessarily a maximal independent broadcast. Consider, for example, the path $P_4 : v_1, v_2, v_3, v_4$ in Figure 1. If we assign $f(v_1) = f(v_4) = 1$ and $f(v_2) = f(v_3) = 0$ (Figure 1(a)), then f is the characteristic function of a maximal independent set, and is an independent broadcast and a minimal dominating broadcast. But this function is not a maximal independent broadcast, as illustrated by the function g defined as follows: $g(v_1) = g(v_4) = 2$, and $g(v_2) = g(v_3) = 0$ (Figure 1(b)).



(a) (b)

Figure 1: Independent broadcasts.

We can see that neither $\gamma(G)$ nor $i(G)$ are comparable with $i_b(G)$ (which we denote by $\gamma(G) \diamond i_b(G)$ and $i(G) \diamond i_b(G)$). For example, it is a simple exercise to show that $\gamma(P_6) =$

$i(P_6) = 2 < 3 = i_b(P_6)$, while for the graph $S(K_{1,t})$, defined earlier, we have $\gamma(S(K_{1,t})) = i(S(K_{1,t})) = t > 2 = i_b(S(K_{1,t}))$. Also, observe that neither $p(G)$ nor $P(G)$ are comparable with $i_b(G)$. For example, the Petersen graph PG has $p(PG) = P(PG) = 1 < 2 = \gamma_b(PG) = i_b(PG)$; while if G is the graph formed from the union of three disjoint copies of P_5 by adding three edges to form a triangle on their centers, then $\gamma_b(G) = i_b(G) = 3 < 5 = p(G) < P(G)$. Thus,

Observation 8 *For any graph G,*

(*i*) $\gamma(G) \leq i(G) \leq \beta_0(G) \leq \beta_b(G)$,
(*ii*) $\{\gamma(G), i(G)\} \diamond i_b(G)$,
(*iii*) $\{p(G), P(G)\} \diamond \{\gamma_b(G), i_b(G)\}$.

Comparisons between the standard domination and independence invariants and the analogous invariants for broadcasts prove to be interesting. Previously we have indicated that

$$\gamma(G) \leq i(G) \leq \beta_0(G) \leq \Gamma(G).$$

This would lead us to wonder if a similar inequality chain holds for the broadcasting invariants:

$$\gamma_b(G) \;??\; i_b(G) \leq \beta_b(G) \;??\; \Gamma_b(G). \tag{2}$$

However, this does not quite prove to be the case. The middle inequality follows immediately from the definition, and the relationship between the first pair of invariants follows from the following. For a vertex $v \in V^+$, let $d^+(v) = min\{d(v,u) \mid u \in V^+ - \{v\}\}$.

Theorem 9 [4] *Let f an independent broadcast on a graph G. If $V^+ = \{v\}$, then f is maximal if and only if $f(v) = e(v)$. On the other hand, if $|V^+| \geq 2$, then f is maximal if and only if the following two conditions are satisfied:*

(*i*) *f is dominating, and,*

(*ii*) *for every $v \in V^+$, $f(v) = d^+(v) - 1$.*

Hence, $\gamma_b(G) \leq i_b(G)$. While a maximal independent broadcast must be dominating, it need not be minimal dominating. For example, Figure 1(b) illustrates a broadcast which is maximal independent but not minimal dominating. The last two invariants in (2), $\beta_b(G)$ and $\Gamma_b(G)$, are incomparable. To see this, consider that $\beta_b(P_4) = 4 > \Gamma_b(P_4) = 3$, but for the Petersen graph PG, we have $\beta_b(PG) = 4 < 5 = \Gamma_b(PG)$. To see that $\beta_b(G)$ and $\Gamma(G)$ are incomparable, observe that $\Gamma(PG) = 5 > 4 = \beta_b(PG)$, while $\beta_b(P_4) = 4 > 2 = \Gamma(P_4)$. Thus,

Corollary 10 *For any graph G,*

(*i*) $\gamma_b(G) \leq i_b(G)$,
(*ii*) $\beta_b(G) \diamond \Gamma_b(G)$, *and*
(*iii*) $\beta_b(G) \diamond \Gamma(G)$.

Next we note the relationship between β_0 and the independent broadcast numbers.

Proposition 11 *For any graph G, $i_b(G) \leq rad(G) \leq \beta_0(G) \leq \beta_b(G)$.*

Proof. The upper bound results from the fact that the characteristic function of an independent set is an independent broadcast. The remainder of the result follows directly from Observation 6 and the fact [5] that $rad(G) \leq \beta_0(G)$. $\square$

Since $\gamma_b(G) \leq i_b(G)$, we have the following corollary.

Corollary 12 *For any graph G, if $\gamma_b(G) = rad(G)$, then $i_b(G) = rad(G)$.*

We note that the converse of Corollary 12 is not true as can be seen with the graph T formed from a subdivided star $S(K_{1,t})$ for $t \geq 3$ and a path P_9 by adding an edge from the center of $S(K_{1,t})$ to an endvertex of P_9. Then it is straightforward to show that $\gamma_b(T) = 5 < 6 = i_b(T) = rad(T)$. Note also that strict inequality is possible in $i_b(G) \leq rad(G)$. For example, $i_b(P_{10}) = 4 < rad(P_{10}) = 5$. To see an i_b-broadcast of P_{10}, consider the assignment f of 0,1,0,1,0,0,1,0,1,0. (Note that every vertex in V^+ is at distance two from another vertex in V^+, f is a maximal independent broadcast. Thus, $i_b(P_{10}) \leq 4$. Since every vertex in V^+ must be within distance $f(v)+1$ of another vertex in V^+, it follows that $i_b(P_{10}) \geq 4$.)

The results in this section and the following inequalities illustrate the relationship between all possible pairs of these eight invariants.

$$\begin{array}{ccccccc} \gamma(G) & \leq & i(G) & \leq & \beta_0(G) & \leq & \Gamma(G) \\ \vee| & & \diamond & & |\wedge & & |\wedge \\ \gamma_b(G) & \leq & i_b(G) & \leq & \beta_b(G) & \diamond & \Gamma_b(G) \end{array}$$

3.3 Independent dominating broadcasts

A broadcast f is called *independent dominating* if it is both independent and dominating. The maximum cost of a minimal independent dominating broadcast of G is called the *upper broadcast independent domination number* and is denoted $\Gamma_{ib}(G)$. Similarly, the *broadcast independent domination number* $\gamma_{ib}(G)$ equals the minimum cost of an independent dominating broadcast of G. Clearly, $\gamma_{ib}(G) \leq i(G)$ and $\Gamma_{ib}(G) \geq \beta_0(G)$, since the characteristic function f_S of any maximal independent set S is a minimal, independent dominating broadcast. Note that if f is a minimal independent dominating broadcast, then for any broadcast $g \neq f$ for which $g \leq f$, we know that g is independent but is not a dominating broadcast.

In domination theory it is well known that a strict inequality $\gamma(G) < i(G)$ often holds. However, for broadcasts we get a different result. We denote by $A \subset B$ that A is a proper subset of B.

Theorem 13 [4] *If f is a broadcast on a graph G that is dominating but not independent, then there is a broadcast g on G that is dominating, independent, with $g(V) \leq f(V)$, and $V_g^+ \subset V_f^+$.*

Corollary 14 [4] *Every graph G has a γ_b-broadcast which is independent, that is, for any graph G,*

$$\gamma_b(G) = \gamma_{ib}(G).$$

Proposition 15 *For any graph G,*

$$\gamma_{ib}(G) \leq i_b(G) \leq rad(G) \leq diam(G) \leq \Gamma_{ib}(G).$$

For the upper broadcast independent domination number, we have the following:

Proposition 16 *For any graph G,*

$$\beta_0(G) \leq \Gamma_{ib}(G) \leq min\{\Gamma_b(G), \beta_b(G)\}.$$

Proof. The lower bound follows from the fact that the characteristic function of a maximal independent set is a minimal independent dominating broadcast. Now let f be a minimal independent dominating broadcast. We must show that f is a minimal dominating broadcast. But if f is independent, then every broadcast g such that $g \neq f$ with $g \leq f$, must be independent. Thus, f must be a minimal dominating broadcast, and hence $\Gamma_{ib}(G) \leq \Gamma_b(G)$. The fact that $\Gamma_{ib}(G) \leq \beta_b(G)$ follows immediately from the fact that every minimal independent dominating broadcast is an independent broadcast. □

On the other hand, we can see that Γ_{ib} and $\Gamma(G)$ are incomparable by considering the Petersen graph PG and the path P_{10}, where $\Gamma_{ib}(PG) = 4 < 5 = \Gamma(PG)$ and $\Gamma(P_{10}) = 5 < 9 = diam(P_{10}) \leq \Gamma_{ib}(P_{10})$.

3.4 Efficient broadcasts

A broadcast f is called *efficient* if every vertex hears exactly one broadcast, that is, for every vertex $v \in V$, $|H(v)| = 1$. The maximum cost of an efficient broadcast is the *upper broadcast efficiency number* $\Gamma_{eb}(G)$, and the *broadcast efficiency number* $\gamma_{eb}(G)$ equals the minimum cost of an efficient broadcast.

For example, Figure 2 illustrates twelve distinct efficient broadcasts in the path P_7, where $\gamma_{eb}(P_7) = 3$ and $\Gamma_{eb}(P_7) = 6$.

In Section 2 it was pointed out that not every graph has an efficient dominating set. However, for broadcasts things are different.

Theorem 17 *Every graph G has a γ_b-broadcast which is efficient.*

Proof. Assume that f is a γ_b-broadcast of G for which $|V^+|$ is a minimum. From Theorem 13, f is independent. Thus assume that f is not efficient, that is, there exists a vertex $v \in V$ with $|H(v)| \geq 2$. Since f is an independent broadcast, $v \in V^0$. Hence there exist two vertices in V^+, say u and w, such that $d(u, v) \leq f(u)$ and $d(v, w) \leq f(w)$. Then there is a path P from u to w of length at most $f(u) + f(w)$. Assume, without loss of generality, that $f(u) \geq f(w)$. Then consider the vertex x on path P at distance $f(w)$ from u.

We can now define a broadcast g as follows:

$g(x) = f(u) + f(w)$;
$g(u) = g(w) = 0$; and
$g(y) = f(y)$, for $y \notin \{x, u, w\}$.

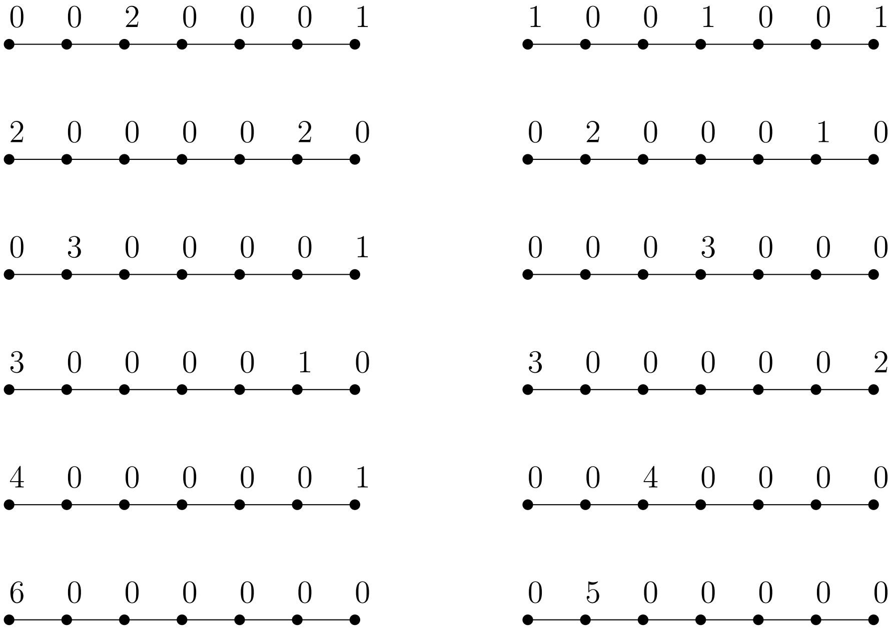


Figure 2: Efficient broadcasts in P_7.

Note that g is a dominating broadcast with cost $g(V) = f(V) = \gamma_b(G)$. Hence, g is a γ_b-broadcast with $|V_g^+| < |V_f^+|$, contradicting our choice of f. □

Corollary 18 *Every graph G has a γ_b-broadcast in which the distance between any two broadcast vertices u and v is greater than $f(u) + f(v)$.*

By definition, $\gamma_b(G) \leq \gamma_{eb}(G)$, so Theorem 17 implies that $\gamma_b(G) = \gamma_{eb}(G)$. We know that $\Gamma_{eb}(G) \leq min\{\beta_b(G), \Gamma_b(G), \Gamma_{ib}(G)\}$ for any graph G, because every efficient dominating broadcast is independent and minimal dominating. Also, observe that any diameter broadcast is an efficient dominating broadcast. Therefore we have the following corollary.

Corollary 19 *For every graph G,*

$$\gamma_b(G) = \gamma_{ib}(G) = \gamma_{eb}(G) \leq i_b(G) \leq rad(G) \leq diam(G) \leq \Gamma_{eb}(G) \leq \Gamma_{ib}(G).$$

We note that neither $p(G)$ nor $P(G)$ is comparable with $\Gamma_{eb}(G)$. For example, for the Petersen graph PG, $p(PG) = P(PG) = 1 < 2 = \Gamma_{eb}(PG)$; while for the complete binary tree T of height six having 63 vertices, it can be verified that $\Gamma_{eb}(T) = 10 < 13 = p(T)$. Also, $\gamma(PG) = 3 > 2 = \Gamma_{eb}(PG)$; while for $n \geq 3$, $\Gamma_{eb}(P_n) = n - 1 > \Gamma(P_n)$. Thus, we make the following observation.

Observation 20 *For graphs G,*

$$\Gamma_{eb}(G) \ \diamond\ \{p(G), P(G), \gamma(G), i(G), \beta_0(G), \Gamma(G)\}.$$

3.5 Packing broadcasts

A broadcast f is called a *packing* if every vertex hears at most one broadcast, that is, for every vertex $v \in V$, $|H(v)| \leq 1$. The maximum cost of a packing broadcast of G is called the *broadcast packing number* and is denoted $P_b(G)$. Similarly, the *lower broadcast packing number* equals the minimum cost of a maximal packing broadcast, and is denoted $p_b(G)$.

Note first that the characteristic function f_S of a maximal packing need not be a maximal packing broadcast. This can be seen by considering the path P_5 with values assigned as indicated in Figure 3(a). This center vertex in Figure 3(a) is a maximal packing, but it is not a maximal packing broadcast, since the value of 1 can be increased to 2, which results in the maximal packing broadcast (see Figure 3(b)).

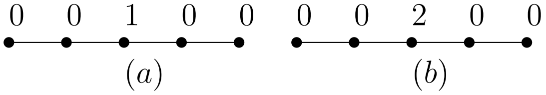


Figure 3: A maximal packing which is not a maximal packing broadcast.

In fact, $p(G)$ and $p_b(G)$ are incomparable. We have just seen, with P_5, a case where $p(G) < p_b(G)$. The reverse inequality holds for the complete binary tree T of height five, having 31 vertices. One can check that a radius broadcast of the tree is a maximal packing broadcast with cost four, while it can be seen that $p(T) = 6$. Note also for the Petersen graph PG, $P_b(PG) = p_b(PG) = 2 > 1 = p(PG) = P(PG)$. Further, any radius or diameter broadcast is a packing broadcast and every packing broadcast is an independent broadcast. Hence, we summarize our observations as follows.

Observation 21 *For any graph G,*

(*i*) $p_b(G) \diamond P(G)$,

(*ii*) $p_b(G) \diamond p(G) \leq P(G) \leq P_b(G)$, *and*

(*iii*) $p_b(G) \leq rad(G) \leq diam(G) \leq P_b(G) \leq \beta_b(G)$.

Note that one may have a maximal packing broadcast which is not a dominating broadcast. For example, consider the values 1,0,0,1,0,0 in the graph of P_6. We know that for any graph G,

$$p(G) \leq P(G) \leq \gamma(G).$$

So again, it is natural to ask if similar inequalities hold between the corresponding broadcast invariants. The answer is 'no'. For example, $P_b(P_6) \geq diam(P_6) = 5$, but $\gamma_b(P_6) = 2$. The complete answer to this question is given next.

Proposition 22 *Every efficient broadcast is*

(*i*) *a maximal packing broadcast,*
(*ii*) *a minimal dominating broadcast, and*
(*iii*) *a minimal independent dominating broadcast.*

Proof. (i) Let f be an efficient broadcast. Then f is, by definition, a packing broadcast. We need to show that f is a maximal packing broadcast. Suppose there exists a packing broadcast g, $g \neq f$ and $g \geq f$. Then there must exist at least one vertex w such that $g(w) > f(w)$. But since, by definition, $g(w) \leq e(w)$, there must be at least one vertex, say x, that hears the broadcast from w in g that does not hear the broadcast from w in f. But since f is an efficient broadcast, x must have heard a broadcast from one vertex in f. This means that x hears at least two broadcasts in g, contradicting our assumption that g is a packing broadcast. Therefore, f is a maximal packing broadcast.

(ii) By definition, every efficient broadcast f is a dominating broadcast. We must show that f is a minimal dominating broadcast. Let $v \in V^+$. Since f is efficient, each vertex of $N_f[v]$ is a private f-neighbor of v. In particular, there exists a vertex u such that $d(u, v) = f(v) = e(v)$, that is, u is a private f-neighbor of v satisfying the minimality condition of Theorem 3. Therefore, if the value $f(v)$ is decreased, then the resulting broadcast will no longer be dominating since vertex u will not hear a broadcast.

(iii) By definition, every efficient broadcast f is an independent dominating broadcast. We must show that f is a minimal independent dominating broadcast. But if f is independent, then every broadcast g such that $g \neq f$ with $g \leq f$, must be independent. Thus, we only need to show that f is a minimal dominating broadcast. But from (ii) above, if f is efficient and dominating, it must be minimal dominating. □

The following corollary is immediate from Corollary 19 and Proposition 22.

Corollary 23 *For any graph G,*

$$p_b(G) \leq \gamma_{eb}(G) = \gamma_{ib}(G) = \gamma_b(G) \leq i_b(G) \leq \Gamma_{eb}(G) \leq min\{P_b(G), \Gamma_b(G), \Gamma_{ib}(G)\}.$$

For the Petersen graph PG, it is straightforward to determine that $2 = P_b(G) < 3 = \gamma(PG) \leq i(PG) \leq \beta_0(PG) \leq \Gamma_{ib}(PG) \leq \Gamma_b(PG) = \Gamma(PG) = 5$. On the other hand, for a path P_n, $\gamma(P_n) = i(P_n) = \lceil n/3 \rceil < \lceil n/2 \rceil = \beta_0(P_n) = \Gamma(P_n) < diam(P_n) = n - 1 \leq P_b(P_n)$. Hence, $P_b(G) \diamond \{\gamma(G), i(G), \beta_0(G), \Gamma(G)\}$.

Also, neither $\Gamma_{ib}(G)$ nor $\Gamma_b(G)$ is comparable with $P_b(G)$. To show this, we need to show that $P_b(G) > \Gamma_b(G) \geq \Gamma_{ib}(G)$ for some graph G. Let S_3 be the tree obtained from the star $K_{1,3}$ by subdividing each edge twice. Form the tree T from three disjoint copies of S_3 by adding a vertex x adjacent to the center of each S_3. We will show that $\Gamma_{eb}(T) = \Gamma_{ib}(T) = \Gamma_b(T) = 20 < 21 \leq P_b(T)$. Let f be a broadcast that assigns values of 3, 2, 2, respectively, to the leaves of each copy of S_3 and 0 to all other vertices as shown in Figure 3.5. Since no values of f can be increased and f still be a packing, it follows f is a maximal packing broadcast. Hence, $P_b(T) \geq f(V) = 21$. The broadcast f is not dominating, but one can attain a dominating broadcast g from this assignment by changing the values to 4,1,1 in exactly one copy of S_3 (See Figure 3.5.) Note that g is an efficient dominating broadcast, so g is a minimal dominating broadcast by Proposition 22. To see that g is a Γ_b-broadcast of T, we must show that $\Gamma_b(T) \leq 20$. NEED TO ADD THIS PROOF HERE.

Hence we have the following observation.

Observation 24 *For graphs G,*

$$P_b(G) \diamond \{\gamma(G), i(G), \beta_0(G), \Gamma(G), \Gamma_{ib}(G), \Gamma_b(G)\}.$$

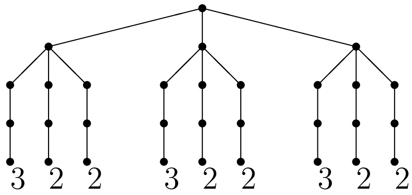


Figure 4: A maximal packing broadcast for T.

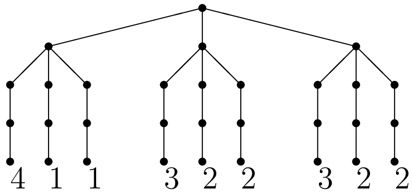


Figure 5: A Γ_b-broadcast for T.

4 Broadcasts in grid graphs

We now consider broadcasts on the class of *grid graphs* $G_{m,n} = P_m \square P_n$, where $2 \leq m \leq n$ and $\square$ denotes the Cartesian product. Let $v_{i,j}$ denote the vertex in row i and column j of $G_{m,n}$. The column to which a vertex x belongs will be denoted $C(x)$.

Lemma 25 *For every integer $n \geq 1$, $\gamma_b(G_{2,n}) = \gamma(G_{2,n}) = rad(G_{2,n})$.*

Proof. Certainly, $\gamma_b(G_{2,n}) \leq rad(G_{2,n}) = \lceil (n+1)/2 \rceil$, and Jacobson and Kinch [8] established that $\gamma(G_{2,n}) = \lceil (n+1)/2 \rceil$, so it suffices to show that $\gamma_b(G_{2,n}) \geq \lceil (n+1)/2 \rceil$. From Theorem 17, $G_{2,n}$ has an efficient γ_b-broadcast, say f. Let $V^+ = \{x_1, x_2, \ldots, x_t\}$, where for each i with $1 \leq i \leq t-1$, $C(x_i) < C(x_{i+1})$. If $t = 1$, then necessarily f is a radius broadcast, so assume that $t \geq 2$. Then x_1 f-dominates at most $4f(x_1) - 1$ vertices, and, similarly, x_t f-dominates at most $4f(x_t) - 1$ vertices. For every integer i with $2 \leq i \leq t-1$, the vertex x_i f-dominates exactly $4f(x_i)$ vertices. Since f is efficient, $4f(x_1) - 1 + 4f(x_t) - 1 + 4[\gamma_b(G_{2,n}) - f(x_1) - f(x_t)] \geq 2n$, giving the required result. $\square$

Corollary 26 *For every positive integer r, there exists a graph G for which $\gamma_b(G) = \gamma(G) = rad(G) = r$.*

As an immediate consequence of Lemma 25, we can determine the broadcast domination number of the 3 by n grid graph.

Lemma 27 *For every integer $n \geq 1$, $\gamma_b(G_{3,n}) = rad(G_{3,n})$.*

Proof. Assume, to the contrary, that $\gamma_b(G_{3,n}) < rad(G_{3,n})$ and let f be a minimum dominating broadcast on $G_{3,n}$. Since f dominates every subgraph of $G_{3,n}$ isomorphic to $G_{2,n}$, it follows that there exists a dominating broadcast g on $G_{2,n}$ with $g(V(G_{2,n})) \leq f(V(G_{3,n}))$. However, $rad(G_{3,n}) = rad(G_{2,n})$, implying that $\gamma_b(G_{2,n}) < rad(G_{2,n})$. Since this contradicts Lemma 25, no such broadcast f exists and the result follows. $\square$

We are now in a position to compute the broadcast domination number of any grid graph.

Theorem 28 *For every pair m, n of integers with $2 \leq m \leq n$,*

$$\gamma_b(G_{m,n}) = rad(G_{m,n}).$$

Proof. The proof is by induction on m. Lemmas 25 and 27 form our basis step. Thus, we assume that $m > 3$ and that the result is true for all graphs $G_{k,n}$ with $2 \leq k < m$. It is easy to see that for $G = G_{m,n}$, $rad(G) = \left\lfloor \frac{m}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor$. Further, we know that $\gamma_b(G) \leq rad(G)$ for any graph G.

Therefore we assume that $\gamma_b(G) < rad(G)$. Then it must hold that $\gamma_b(G) \leq \left\lfloor \frac{m}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor - 1 = \left\lfloor \frac{m-2}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor$. The last quantity here is the radius of any grid graph $G_{m-2,n}$. Thus, let H be the subgraph induced in G by the first $m-2$ rows of G. By our inductive hypothesis, we know that $rad(H) = \gamma_b(H)$. Thus, $\gamma_b(G) \leq rad(H) = \gamma_b(H)$.

It is straightforward to see at this point that this must mean that $\gamma_b(G) = \gamma_b(H)$. For if not, we could consider a γ_b-broadcast f of G. Then for any vertex $v_{k,i}$ with $k \in \{m-1, m\}$ and for all $i = 1, 2, \ldots, n$, adding the value $f(v_{k,i})$ to the value $f(v_{m-2,i})$ creates a dominating broadcast of H with cost less than $\gamma_b(H)$, a contradiction.

Case 1. There exists a γ_b-broadcast f of G which has some positive value assigned to a vertex, say $v_{k,i}$, in one of the last two rows. Then we define a broadcast g on H in the following way. We assign the value $f(v_{m-2,i}) + f(v_{m-1,i}) + f(v_{m,i}) - 1$ to $g(v_{m-2,i})$. For all $j \neq i$ where $1 \leq j \leq n$, let $g(v_{m-2,j}) = f(v_{m-2,j}) + f(v_{m-1,j}) + f(v_{m,j})$. And for any vertex in the first $m-3$ rows, let $g(v) = f(v)$. Clearly the function g is a dominating broadcast of H with cost $\gamma_b(G) - 1 < \gamma_b(H)$, a contradiction.

Case 2. No γ_b-broadcast of G has vertices from the last two rows of G in V^+. Let f be a γ_b-broadcast of G and let j be the maximum value such that $v_{j,i} \in V^+$. Then $j \leq m-2$, and we consider the function g created by assigning the value $f(v_{j-1,i}) + f(v_{j,i}) - 1$ to $g(v_{j-1,i})$, $g(v_{j,i}) = 0$, and for all other vertices we allow $g(v) = f(v)$. If g is a dominating broadcast for H, then its cost of $\gamma_b(H) - 1$ would yield a contradiction. Thus, there must be a vertex $v_{k,l}$ with $j \leq k \leq m-2$ which was dominated by f but is not dominated by g. Hence, $d(v_{j,i}, v_{k,l}) \in \{f(v_{j,i}), f(v_{j,i}) - 1\}$ and $v_{k,l}$ is a private f-neighbor of $v_{j,i}$. But then $v_{m,l}$ cannot be in $N_f[v]$ for any $v \in V_f^+$, again a contradiction.

Since we are led to a contradiction in every case, it must be true that $\gamma_b(G) = rad(G)$. $\Box$

5 Concluding Remarks

We conclude with a summary (see Table 1) of the relationships among the parameters discussed in this paper and a list of open problems.

Let r and d denote the radius and diameter, respectively. In Table 1, if the relation R occurs in row A and column B, this should be read as ARB. For example, reading across row 1, we have $p(G) = p(G)$, $p(G) \leq P(G)$, and so on.

Among the many open problems raised by our initial study of broadcasts in graphs, and

Table 1: Broadcast parameters inequalities.

	p	P	γ	i	β_0	Γ	γ_b	i_b	β_b	Γ_b	r	d	Γ_{ib}	Γ_{eb}	p_b	P_b
p	$=$	$\leq$	$\leq$	$\leq$	$\leq$	$\leq$	$\diamond$	$\diamond$	$\leq$	$\leq$	$\diamond$	$\diamond$	$\leq$	$\diamond$	$\diamond$	$\leq$
P	$\geq$	$=$	$\leq$	$\leq$	$\leq$	$\leq$	$\diamond$	$\diamond$	$\leq$	$\leq$	$\diamond$	$\diamond$	$\leq$	$\diamond$	$\diamond$	$\leq$
γ	$\geq$	$\geq$	$=$	$\leq$	$\leq$	$\leq$	$\geq$	$\diamond$	$\leq$	$\leq$	$\diamond$	$\diamond$	$\leq$	$\diamond$	$\geq$	$\diamond$
i	$\geq$	$\geq$	$\geq$	$=$	$\leq$	$\leq$	$\geq$	$\diamond$	$\leq$	$\leq$	$\diamond$	$\diamond$	$\leq$	$\diamond$	$\geq$	$\diamond$
β_0	$\geq$	$\geq$	$\geq$	$\geq$	$=$	$\leq$	$\geq$	$\geq$	$\leq$	$\leq$	$\geq$	$\diamond$	$\leq$	$\diamond$	$\geq$	$\diamond$
Γ	$\geq$	$\geq$	$\geq$	$\geq$	$\geq$	$=$	$\geq$	$\geq$	$\diamond$	$\leq$	$\geq$	$\diamond$	$\diamond$	$\diamond$	$\geq$	$\diamond$
γ_b	$\diamond$	$\diamond$	$\leq$	$\leq$	$\leq$	$\leq$	$=$	$\leq$	$\leq$	$\leq$	$\leq$	$\leq$	$\leq$	$\leq$	$\geq$	$\leq$
i_b	$\diamond$	$\diamond$	$\diamond$	$\diamond$	$\leq$	$\leq$	$\geq$	$=$	$\leq$	$\leq$	$\leq$	$\leq$	$\leq$	$\leq$	$\geq$	$\leq$
β_b	$\geq$	$\geq$	$\geq$	$\geq$	$\geq$	$\diamond$	$\geq$	$\geq$	$=$	$\diamond$	$\geq$	$\geq$	$\geq$	$\geq$	$\geq$	$\geq$
Γ_b	$\geq$	$\geq$	$\geq$	$\geq$	$\geq$	$\geq$	$\geq$	$\geq$	$\diamond$	$=$	$\geq$	$\geq$	$\geq$	$\geq$	$\geq$	$\diamond$
r	$\diamond$	$\diamond$	$\diamond$	$\diamond$	$\leq$	$\leq$	$\geq$	$\geq$	$\leq$	$\leq$	$=$	$\leq$	$\leq$	$\leq$	$\geq$	$\leq$
d	$\diamond$	$\diamond$	$\diamond$	$\diamond$	$\diamond$	$\diamond$	$\geq$	$\geq$	$\leq$	$\leq$	$\geq$	$=$	$\leq$	$\leq$	$\geq$	$\leq$
Γ_{ib}	$\geq$	$\geq$	$\geq$	$\geq$	$\geq$	$\diamond$	$\geq$	$\geq$	$\leq$	$\leq$	$\geq$	$\geq$	$=$	$\geq$	$\geq$	$\diamond$
Γ_{eb}	$\diamond$	$\diamond$	$\diamond$	$\diamond$	$\diamond$	$\diamond$	$\geq$	$\geq$	$\leq$	$\leq$	$\geq$	$\geq$	$\leq$	$=$	$\geq$	$\leq$
p_b	$\diamond$	$\diamond$	$\leq$	$\leq$	$\leq$	$\leq$	$\leq$	$\leq$	$\leq$	$\leq$	$\leq$	$\leq$	$\leq$	$\leq$	$=$	$\leq$
P_b	$\geq$	$\geq$	$\diamond$	$\diamond$	$\diamond$	$\diamond$	$\geq$	$\geq$	$\leq$	$\diamond$	$\geq$	$\geq$	$\diamond$	$\geq$	$\geq$	$=$

in addition to those raised in earlier sections, the following are of particular interest to the authors.

1. For which graphs G does $\gamma_b(G) = \gamma(G)$? Call these graphs of Type I.
2. For which graphs G does $\gamma_b(G) = rad(G)$? Call these graphs of Type II.
3. For which graphs G is $\gamma_b(G) < min\{\gamma(G), rad(G)\}$? Call these graphs of Type III.
4. Can you characterize trees of Type I? Type II? or Type III?
5. What can you say about the class of minimum cost dominating broadcasts, where the number of broadcast vertices is a minimum (or a maximum)?
6. Can you construct linear algorithms for computing the values of each of the broadcasting invariants for trees?
7. Can you settle the complexity of the decision problems associated with each of the broadcasting invariants?
8. What are the values of each of the broadcasting invariants for an $m \times n$ grid graph?
9. Under what conditions is $\gamma_b(G) = i_b(G)$? (See Corollary 10).
10. Can you develop Nordhaus-Gaddum bounds [11] for the broadcasting invariants?

11. Suppose you are allowed to assign only broadcast powers of 0, 1 or 2 to the vertices of a graph. This suggests the concept of the broadcast domination number with limited broadcast power, say indexed by k, which could give rise to the *k-limited broadcast domination number* $\gamma_{kb}(G)$. What can you say about this invariant?

References

[1] D. W. Bange, A. E. Barkauskas, and P. J. Slater, Efficient dominating sets in graphs. In *Applications of Discrete Mathematics*, R. D. Ringeisen and F. S. Roberts, Eds., SIAM (1988) 189–199.

[2] C. Berge, *Theory of Graphs and its Applications*, Collection Universitaire de Mathématiques, No. 2, Dunod, Paris, 1958.

[3] D. Erwin, Dominating broadcasts in graphs. Submitted.

[4] D. Erwin, Independent broadcasts in graphs. Preprint.

[5] O. Favaron, M. Mahéo, and J. F. Saclé, Some results on conjectures of Graffiti-I. *Ars Combin.* 29C (1990) 90–106.

[6] T. W. Haynes, S. T. Hedetniemi, and P. J. Slater, *Fundamentals of Domination in Graphs*, Marcel Dekker, Inc. New York, 1998.

[7] T. W. Haynes, S. T. Hedetniemi, and P. J. Slater, editors, *Domination in Graphs: Advanced Topices*, Marcel Dekker, New York, 1998.

[8] M. S. Jacobson and L. F. Kinch, On the domination number of products of graphs: I. *Ars Combin.* 18 (1984) 33–44.

[9] C. L. Liu, *Introduction to Combinatorial Mathematics*, McGraw-Hill, New York, NY (1968).

[10] M. A. Henning, Distance domination in graphs. In *Domination in Graphs: Advanced Topics*, T.W. Haynes, S.T. Hedetniemi and P.J. Slater, Eds., Marcel Dekker, New York (1998) 321–349.

[11] E. A. Nordhaus and J. W. Gaddum, On complementary graphs. *Amer. Math. Monthly* 63 (1956) 175–177.

[12] Ore, *Theory of Graphs*, Amer. Math. Soc. Publ., 38 (Amer. Math. Soc., Providence, RI), 1962.

[13] P. J. Slater, *R*-domination in graphs. *J. Assoc. Comput. Mach.* 23 (1976) 446–450.