Differentials in Graphs

[1]J.R. Lewis, [1]T.W. Haynes, [2]S.M. Hedetniemi, [2]S.T. Hedetniemi, and [3]P.J. Slater

[1]Department of Mathematics
East Tennessee State University
Johnson City, TN 37614 USA

[2] Department of Computer Science
Clemson University
Clemson, SC 29634 USA

[3]Mathematical Sciences and Computer Science Departments
University of Alabama in Huntsville
Huntsville, AL 35899 US

Dedicated to Professor Henda Swart

Abstract

Let $G = (V, E)$ be an arbitrary graph, and consider the following game. You are allowed to buy as many tokens as you like, say k tokens, at a cost of \$1 each. You then place the tokens on some subset of k vertices of V. For each vertex of G which has no token on it, but is adjacent to a vertex with a token on it, you receive \$1. Your objective is to maximize your profit, that is, the total value received minus the cost of the tokens bought. Let $\mathrm{B}(X)$ be the set of vertices in $V - X$ that have a neighbor in a set X. Based on this game, we define the *differential* of a set X to be $\partial(X) = |\mathrm{B}(X)| - |X|$, and the *differential of a graph* to equal the $\max\{\partial(X)\}$ for any subset X of V. In this paper, we introduce several different variations of the differential of a graph and study bounds on, and properties of, these novel parameters.

Keywords: differential, domination, profit margin of a graph

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1 Introduction

Let $G = (V, E)$ be a graph with no isolated vertices. Consider a game where you are allowed to buy as many tokens as you like, at a cost of \$1 each. For example, suppose that you buy k tokens. You then place the tokens on some subset of k vertices of G. For each vertex of G which has no token on it, but is adjacent to a vertex with a token on it, you receive \$1. Your objective is to maximize your profit, that is, the total value received minus the cost of the tokens bought. Notice that you do not receive any credit for the vertices on which you place a token.

For a vertex $v \in V$, the *open neighborhood* of v is the set $N(v) = \{u \in V \mid uv \in E\}$ and the *closed neighborhood* is the set $N[v] = N(v) \cup \{v\}$. For a set $S \subseteq V$, its *open neighborhood* is $N(S) = \cup_{v \in S} N(v)$ and its *closed neighborhood* is $N[S] = N(S) \cup S$. The subgraph induced by S is denoted by $\langle S \rangle$.

For a set $X \subseteq V$, we define:
$I(X) = X - N(X)$, the isolates in $\langle X \rangle$, the vertices in X having no neighbors in X,
$A(X) = X \cap N(X)$, the non-isolates in $\langle X \rangle$, the vertices in X having a neighbor in X,
$B(X) = (V - X) \cap N(X)$, the *boundary* of X, the vertices in $V - X$ dominated by X.
Notice that $I(X) \cap A(X) = \emptyset$ and $X = I(X) \cup A(X)$.

Intuitively, every vertex in $V - X$ which is dominated by a vertex in X gives X a differential of $+1$, while each vertex in X creates a negative differential of -1.

We are now ready to define several 'differentials' of a set X.
The *I-differential* of a set X is $\partial_I(X) = |\mathrm{B}(X)| - |I(X)|$.
The *A-differential* of a set X is $\partial_A(X) = |\mathrm{B}(X)| - |A(X)|$.
The *B-differential* of a set X is $|B(X)|$.
The *differential* of a set X is $\partial(X) = |\mathrm{B}(X)| - |X|$.

The definition of the A-differential of a set was first given by McRae and Parks [5], while the definition of $\partial(X)$ was given by Hedetniemi about ten years ago [4]. The parameter $\partial(X)$ is also considered by Goddard and Henning [2], who denoted it $\eta(X)$. The minimum differential of an independent set has been considered by Zhang [8], who showed that this parameter can be computed in polynomial time.

Clearly, some sets have a positive differential, some sets have a negative differential and some sets have zero differential. Consider therefore the collection of differentials of all subsets of vertices of a graph G.

Extending from a set to a graph, we define the following invariants.
The *I-differential* of a graph is $\partial_I(G) = max\{\partial_I(X) \mid X \subseteq V\}$.
The *A-differential* of a graph is $\partial_A(G) = max\{\partial_A(X) \mid X \subseteq V\}$.
The *B-differential* of a graph is $\Psi(G) = max\{|\mathrm{B}(X)| \mid X \subseteq V\}$.
The *differential* of a graph is $\partial(G) = max\{\partial(X) \mid X \subseteq V\}$.

The parameter $\Psi(G)$ was introduced by Slater in [6] and is called the *enclaveless number*.

We consider differentials of arbitrary graphs in Section 2 and differentials of trees in Section 3, but we first give a few definitions. In general we will follow the notation and terminology of [3]. Given a set $S \subseteq V$ the *private neighborhood* $\mathrm{pn}[v, S]$ of $v \in S$ is defined by $\mathrm{pn}[v, S] = N[v] - N[S - \{v\}]$, equivalently, $\mathrm{pn}[v, S] = \{u \in V \mid N[u] \cap S = \{v\}\}$. Each vertex in $\mathrm{pn}[v, S]$ is called a *private neighbor* of v with respect to S. The *external private neighborhood* $\mathrm{epn}(v, S)$ of v with respect to S consists of those private neighbors of v in $V - S$. Thus, $\mathrm{epn}(v, S) = \mathrm{pn}[v, S] \cap (V - S)$.

A set $S \subseteq V$ is a *dominating set* if $N[S] = V$. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set. We call a dominating set of G of minimum cardinality a γ*-set*. We use similar notation for other parameters, that is, for a generic parameter $\mu(G)$, we call a set satisfying the property for the parameter and having cardinality $\mu(G)$, a μ-set. A set $S \subseteq V$ is a *total dominating set* if $N(S) = V$. The *total domination number* $\gamma_t(G)$ is the minimum cardinality of a minimal total dominating set. A set S is *independent* if no two vertices in S are adjacent. The *independent domination number* $i(G)$ equals the minimum cardinality of an independent dominating set.

2 Differentials in Arbitrary Graphs

Our observations follow directly from the definitions.

Proposition 1 *For any graph G,*

(i) $\partial(G) \leqslant \partial_A(G) \leqslant \Psi(G)$.

(ii) $\partial(G) \leqslant \partial_I(G) \leqslant \Psi(G)$.

The differential values for paths and cycles are straightforward to determine; we state them without proof.

Proposition 2 *For paths P_n, $n \geqslant 1$ and cycles C_n, $n \geqslant 3$,*

(i) $\partial_I(P_n) = \partial_I(C_n) = \left\lfloor \frac{n}{2} \right\rfloor$.

(ii) $\partial_A(P_n) = \partial_A(C_n) = 2\left\lfloor \frac{n}{3} \right\rfloor$.

(iii) $\Psi(P_n) = \Psi(C_n) = 2\left\lfloor \frac{n}{3} \right\rfloor$.

(iv) $\partial(P_n) = \partial(C_n) = \left\lfloor \frac{n}{3} \right\rfloor$.

In fact the value $\Psi(G)$ in G can be determined in terms of $\gamma(G)$ for all graphs G.

Proposition 3 [6] *For any graph G of order n, $\Psi(G) = n - \gamma(G)$.*

Consequently the following problem is NP-complete, even for bipartite and chordal graphs ([3]).

BOUNDARY
INSTANCE: Graph $G = (V, E)$, positive integer k
QUESTION: Does G have a set $X \subseteq V(G)$ such that $|\mathrm{B}(X)| \geqslant k$?

Proposition 4 *For any graph G of order n without isolated vertices,*

(i) $\Psi(G) - \gamma(G) = n - 2\gamma(G) \leqslant \partial(G) \leqslant n - \gamma(G) - 1 = \Psi(G) - 1$,

(ii) $n - \gamma_t(G) \leqslant \partial_I(G)$,

(iii) $n - 2\gamma_t(G) \leqslant \partial_A(G)$, and

(iv) $n - i(G) \leqslant \partial_A(G)$.

Proof:

(i) Let X be a γ-set. Notice that,

$$\partial(G) \geqslant \partial(X) = |\mathrm{B}(X)| - |X| = (n - \gamma(G)) - \gamma(G) = n - 2\gamma(G).$$

Let X be a ∂-set. Then it follows from Proposition 3 that

$$\begin{aligned} \partial(G) = \partial(X) &= |\mathrm{B}(X)| - |X| \\ &\leqslant \Psi(G) - |X| \\ &= n - \gamma(G) - |X|. \end{aligned}$$

Since $|X| \geqslant 1$, the result holds.

(ii) Let X be a γ_t-set. We note that $|\mathrm{B}(X)| = n - \gamma_t(G)$. Thus,

$$\begin{aligned} \partial_I(G) \geqslant \partial_I(X) &= |\mathrm{B}(X)| - |I(X)| \\ &= n - \gamma_t(G) - 0. \end{aligned}$$

(iii) Furthermore,

$$\begin{aligned} n - 2\gamma_t(G) &\leqslant n - 2\gamma(G) \\ &\leqslant \partial(G) \\ &\leqslant \partial_A(G). \end{aligned}$$

(iv) Let X be an i-set. Then $|\mathrm{B}(X)| = n - i(G)$. Thus

$$\begin{aligned} \partial_A(G) \geqslant \partial_A(X) &= |\mathrm{B}(X)| - |A(X)| \\ &= n - i(G) - 0. \end{aligned}$$

Proposition 5 *For any graph G with maximum degree $\Delta(G)$,*

(i) $\Delta(G) - 1 \leqslant \partial(G)$,

(ii) $\Delta(G) \leqslant \partial_A(G)$.

Proof: Let $X = \{v\}$, where v is a vertex of maximum degree $\Delta(G)$. We have

$$\begin{aligned} \partial(G) \geqslant \partial(X) &= |\mathrm{B}(X)| - |X| \\ &= \Delta(G) - 1, \text{ and} \end{aligned}$$

$$\begin{aligned} \partial_A(G) \geqslant \partial_A(X) &= |\mathrm{B}(X)| - |A(X)| \\ &= \Delta(G) - 0 \\ &= \Delta(G). \quad \square \end{aligned}$$

The first degree sequence bound for a domination parameter appears in [7]. Namely, if graph G has degree sequence $(d_1, d_2, \dots, d_n)$ with $d_i \geqslant d_{i+1}$ for $1 \leqslant i \leqslant n-1$, then the domination number satisfies $\gamma(G) \geqslant \ min\{k : (d_1 + d_2 + \dots + d_k) + k \geqslant n\}$. A similar (in this case, lower) bound applies for the differential, namely, the following.

Theorem 6 *Assume G has degree sequence $(d_1, d_2, \dots, d_n)$ with $d_1 \geqslant d_2 \geqslant \dots \geqslant d_n \geqslant 1$. Let $t = min\{k : (d_1 + d_2 + ... + d_k) + k \geqslant n\}$. Then $\partial(G) \leqslant (d_1 + d_2 + \dots + d_t) - t$.*

Proof: Suppose that $X \subset V(G)$ and $\partial(X) = \partial(G)$. Let $X = \{x_1, x_2, \dots, x_j\}$, and note that $|\mathrm{B}(X)| \leqslant deg(x_1) + deg(x_2) + \dots + deg(x_j) \leqslant d_1 + d_2 + \dots + d_j$ and that $\partial(X) \leqslant (deg(x_1) + deg(x_2) + \dots + deg(x_j)) - j \leqslant (d_1 + d_2 + \dots + d_j) - j$. If $j \leqslant t$, then $\partial(X) = \partial(G) \leqslant (d_1 + d_2 + \dots + d_j) - j \leqslant (d_1 + d_2 + \dots + d_j) - j + (d_{j+1} - 1) + \dots + (d_t - 1)$, as required. Assuming that $j \geqslant t$, we have $|\mathrm{B}(X)| \leqslant |V(G) - X| \leqslant n - j$ and $\partial(X) \leqslant (n - j) - j = n - 2j$. Also, $(d_1 + d_2 + \dots + d_t) + t \geqslant n$, so $(d_1 + d_2 + \dots + d_t) - t \geqslant n - 2t \geqslant n - 2j \geqslant \partial(X) = \partial(G)$, completing the proof. □

Corollary 7 *If G is r-regular and $n = (r + 1)k$, then $\partial(G) \leqslant k(r - 1)$ and $\partial(G) = k(r - 1)$ if and only if G has an efficient dominating set.*

A set $S \subseteq V(G)$ is a *2-packing* if for each pair of vertices $u, v \in S$, the $N[u] \cap N[v] = \emptyset$. The *packing number* $\rho(G)$ is the cardinality of a maximum 2-packing.

Proposition 8 *For every k-regular graph G,*

(i) $\partial(G) \geqslant \rho(G)(k - 1)$,

(ii) $\partial_A(G) \geqslant \rho(G)k$.

Proof: Let X be a ρ-set. Since G is k-regular, $|\mathrm{B}(X)| = \rho(G)k$ and hence,

$$\partial(G) \geqslant \partial(X) = |\mathrm{B}(X)| - |X| = \rho(G)k - \rho(G) = \rho(G)(k - 1), \text{ and}$$

$$\partial_A(G) \geqslant \partial(X) = |\mathrm{B}(X)| - |A(X)| = \rho(G)k - 0 = \rho(G)k. \qquad \square$$

3 Differentials in Trees

We have seen from Proposition 4(i) and 5 that for any graph G,

$$n - 2\gamma(G) \leq \partial(G) \leq n - \gamma(G) - 1, \text{ and}$$
$$\Delta(G) - 1 \leq \partial(G).$$

In Section 3.1, we consider realizability of the differential values from $n - 2\gamma(T)$ to $n - \gamma(T) - 1$ for trees T. In Section 3.2, we characterize the trees T that attain the upper bound of $n - \gamma(T) - 1$, and in Section 3.3 we investigate the trees T that attain the lower bound of $n - 2\gamma(T)$. Finally in Section 3.4, we give a characterization of the trees T having $\partial(T) = \Delta(T) - 1$. A vertex of degree one is called a *leaf* and its neighbor is a *support vertex*.

3.1 Realizability

From Proposition 4i, we have $n - 2\gamma(G) \leqslant \partial(G) \leqslant n - \gamma(G) - 1$. Next we show that there exists a tree T having $\gamma(T) \leqslant \partial(T) = k$ for each k, $n - 2\gamma(T) \leqslant k \leqslant n - \gamma(T) - 1$.

Theorem 9 *For any triple (a, b, c) of positive integers such that $a \leqslant b \leqslant c$ and $c - 2a \leqslant b \leqslant c - a - 1$, there exists a tree T having order $n = c$, $\gamma(T) = a$, and $\partial(T) = b$.*

Proof: Let a, b, and c be integers where $a \leqslant b \leqslant c$ and $c - 2a \leqslant b \leqslant c - a - 1$, we will show that the tree T illustrated in Figure 1 has order $n = c$, $\gamma(T) = a$, and $\partial(T) = b$.

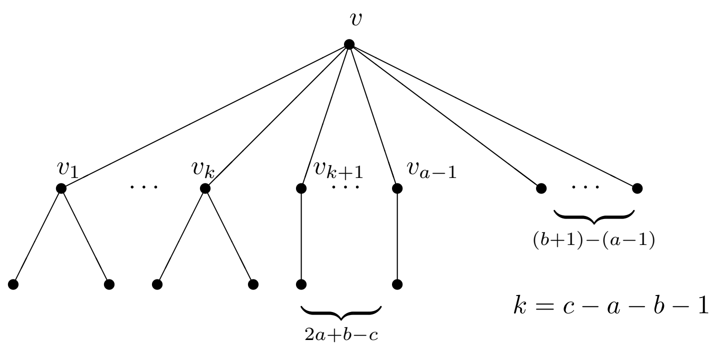

Figure 1: A tree T with $n = c$, $\gamma(T) = a$, and $\partial(T) = b$.

Notice that $\deg(v) = b + 1$ and excluding v, there are $a - 1$ support vertices. It is straightforward to show that $\gamma(T) = a$ and $n = c$.

Next we will show that $\partial(T) = b$. Now $\partial(T) \geqslant \partial(\{v\}) = deg(v) - 1 = b$. To see that $\partial(T) \leqslant b$, let D be a ∂-set. If $v \in D$, then we may assume that no leaf is in D and no vertex of $N(v)$ is in D (if one was the net effect on the differential is at most 0). Therefore, $D = \{v\}$ and $\partial(D) = \partial(\{v\}) = b$. If $v \notin D$, then since $(b + 1) - (a - 1) \geqslant 2$, it follows that $\partial(D \cup \{v\}) > \partial(D) = \partial(T)$, a contradiction. Thus, $\partial(T) = b$ as desired. $\square$

3.2 Trees T with $\partial(T) = n - \gamma(T) - 1$

Domke, Dunbar, and Markus [1] characterized the trees T for which $\Delta(T) = n - \gamma(T)$. We shall show that these trees are precisely the trees obtaining the upper bound of Proposition 4(i). A *subdivision* of an edge uv is obtained by removing edge uv, adding a new vertex w, and adding edges uw and vw. A *wounded spider* is the graph formed by subdividing at most $t-1$ of the edges of a star $K_{1,t}$ for $t \geq 0$. Note that a star is a wounded spider. See Figure 2 for another example.

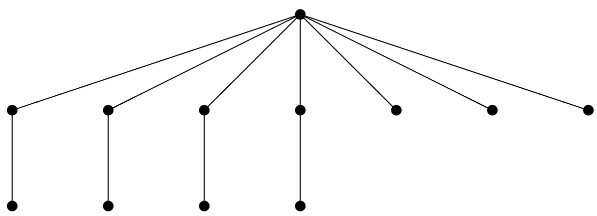

Figure 2: A wounded spider.

Theorem 10 [1] *For a tree T, $\gamma(T) = n - \Delta(T)$ if and only if T is a wounded spider.*

Theorem 11 *A tree T has $\partial(T) = n - \gamma(T) - 1$ if and only if T is a nontrivial wounded spider.*

Proof: Assume that T has $\partial(T) = n - \gamma(T) - 1$, and let X be a ∂-set. Hence, $n - \gamma(T) - 1 = \partial(T) = \partial(X) = |\mathrm{B}(X)| - |X| \leq \Psi(T) - |X| = n - \gamma(T) - |X| \leq n - \gamma(T) - 1$. Thus, equality applies throughout implying that $|X| = 1$. Then $X = \{x\}$ and $deg(x) = |\mathrm{B}(X)| = n - \gamma(T)$. If $deg(x) < \Delta(T)$, then $\partial(\{y\}) > \partial(X)$ where y is a vertex of maximum degree, a contradiction. Hence, $deg(x) = \Delta(T) = n - \gamma(T)$, and these trees are precisely the trees characterized in Theorem 10 (with the exception of the trivial tree which we exclude). □

3.3 Trees T with $\partial(T) = n - 2\gamma(T)$

We next consider trees that achieve the lower bound of Proposition 4(i).

A graph G has *property $\mathcal{EPN}$* if for every γ-set S and for every $v \in S$, $\mathrm{epn}(v, S) \neq \emptyset$. We call a tree with property $\mathcal{EPN}$ an $\mathcal{EPN}$-tree.

Lemma 12 *If G does not have property $\mathcal{EPN}$, then $\partial(G) \geqslant n - 2\gamma(G) + 1$.*

Proof: If G has a γ-set S and $u \in S$ such that $\mathrm{epn}(u, S) = \emptyset$, then

$$\begin{array}{rcl} \partial(G) & \geqslant & \partial(S - \{u\}) \\ & = & n - \gamma(G) - (\gamma(G) - 1) \\ & = & n - 2\gamma(G) + 1. \quad \square \end{array}$$

The tree T in Figure 3 is an example of an $\mathcal{EPN}$-tree. Clearly, $\gamma(T) = 2$ and $\partial(T) = \partial(\{u, v\}) = 4 = n - 2\gamma(T)$.

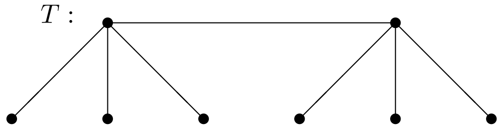

Figure 3: An $\mathcal{EPN}$-tree T.

Notice that $\gamma(P_{3k}) = 2k$ and $\partial(P_{3k}) = k$. Furthermore, P_{3k} is an $\mathcal{EPN}$-tree.

However, the converse of Lemma 12 is not true, as can be seen with the subdivided star $K^*_{1,t}$, $t \geqslant 3$, where $n = 2t + 1$, $\gamma(K^*_{1,t}) = t$, and $\partial(K^*_{1,t}) = t - 1$.

We are continuing to work on the problem of characterizing the trees T for which $\partial(T) = n - 2\gamma(T)$.

3.4 Trees T with $\partial(T) = \Delta(T) - 1$

Finally we characterize the trees having $\partial(T) = \Delta(T) - 1$. For a rooted tree T, let T_u denote the subtree of T induced by u and its descendents. Let P_n denote a path on n vertices.

We define a family $\mathcal{T}$ of trees T described as follows: T is a tree rooted at a vertex v of maximum degree $\Delta(T)$ and one of the following properties holds:

(i) v is adjacent to exactly one leaf x and for each $u \in N(v) - \{x\}$, $T_u \in \{P_2, P_3\}$, where u is an endvertex of T_u, or

(ii) There exist two vertices $x, y \in N(v)$ such that $T_x \in \{P_1, P_2,\}$ and $T_y \in \{P_1, P_2\}$. And for each $u \in N(v) - \{x, y\}$, the subtree $T_u \in \{P_1, P_2, P_3, P_4, P_5\}$, where u is the center of T_u or u is a leaf of $T_u = P_3$.

For examples, see Figures 4 and 5.

Theorem 13 *A tree T has $\partial(T) = \Delta(T) - 1$ if and only if $T \in \mathcal{T}$.*

Proof: Clearly, $\Delta(T) \geqslant 1$ and the theorem holds if $\Delta(T) \leqslant 2$. Thus assume that $\Delta(T) \geqslant 3$. Let $\partial(T) = \Delta(T) - 1$, and let v be a vertex of maximum degree in T. Note that $D = \{v\}$ is a ∂-set. If any vertex $x \in V - N[v]$ has at least two neighbors in $V - N[v]$, then $\partial(\{v, x\}) \geqslant \Delta(T) + 2 - 2 > \Delta(T) - 1 = \partial(T)$, a contradiction. Hence, $V - N[v]$ induces $rK_1 \cup sK_2$ where $r \geqslant 0$ and $s \geqslant 0$. Let $H_1 = rK_1$ and $H_2 = sK_2$.

If any vertex $u \in N(v)$ has three or more neighbors in $V - N[v]$, then $\partial(\{v, u\}) \geqslant \Delta(T) - 1 + 3 - 2 > \Delta(T) - 1 = \partial(T)$, a contradiction. Moreover, since T is a tree, $N(v)$ is an independent set. Hence, $\deg(u) \leqslant 3$ for all $u \in N(v)$.

Furthermore,

$$\begin{array}{rcc} \partial\big(N(v)\big) & = & \sum_{u\in N(v)}(\deg(u)-1)+1-|N(v)| \\ & = & \sum_{u\in N(v)}(\deg(u)-1)+1-\Delta(T) \\ & \leqslant & \partial(T)=\Delta(T)-1. \end{array}$$

This implies that at least two vertices in $N(v)$ have degree at most two. Among all vertices of $N(v)$ with degree at most two, select two, say x and y, such that priority is given first to leaves and next to support vertices. If each of x and y is a leaf or a support vertex, then we have shown that $T \in \mathcal{T}$. Hence assume that this is not the case. Thus, without loss of generality, y has a neighbor in $V(H_2)$. Label the vertices of $N(v) - \{x, y\}$, u_i for $1 \leqslant i \leqslant \Delta(T) - 2$. Thus by our choice of x and y, every vertex u_i in $N(v) - \{x, y\}$ has degree three or has exactly one neighbor, say w_i, in $V(H_2)$.

Let $U = \{u_i \mid \deg(u_i) = 3\}$ and $W = \{w_i \mid w_i \in V(H_2) \cap N(u_i) \text{ and } \deg(u_i) = 2\}$. Let y' be the neighbor of y in $V(H_2)$.

If $U \neq \emptyset$, then

$$\begin{array}{rcc} \partial(T) & \geqslant & \partial(U\cup W\cup\{y'\}) \\ & = & 1+2|U|+2|W|+2-|U|-|W|-1 \\ & = & |U|+|W|+2 \\ & = & \Delta(T) \\ & > & \partial(T), \end{array}$$

a contradiction.

Hence, $U = \emptyset$, that is, $\deg(u_i) \leqslant 2$ in T and $T_{u_i} = P_3$ for all $u_i \in N(v)$.

If x is not a leaf, then $T_x \in \{P_2, P_3\}$ and x is a leaf of T_x. It follows that

$$\begin{array}{rcc} \partial(T) & \geqslant & \partial(W\cup\{x,y'\}) \\ & \geqslant & 2|W|+2+2-|W|-2 \\ & = & \Delta(T) \\ & > & \partial(T), \end{array}$$

a contradiction.

Hence, x is a leaf and T has property (i). Thus, $T \in \mathcal{T}$.

Assume that $T \in \mathcal{T}$. Then $\partial(T) \geqslant \partial(\{v\}) = \Delta(T) - 1$. To see that $\partial(T) \leqslant \Delta(T) - 1$, let D be a ∂-set. Assume first that $v \in D$. Since property (i) or (ii) holds, adding another vertex to D does not increase the differential of T. Hence, we may assume that $D = \{v\}$ and $\partial(T) = \Delta(T) - 1$. Suppose then that $v \notin D$. If property (i) holds, then $N(v)$ contains exactly one leaf x and $\deg(u) = 2$ for each $u \in N(v) - \{x\}$. Hence at most $\Delta(T) - 1$ vertices can contribute one to the differential of T implying that $\partial(T) \leqslant \Delta(T) - 1$. If property (ii) holds, then $N(v)$ contains at least two vertices, say x and y, each of which is a leaf or a support vertex of degree two. Then $\partial(\{x, y\}) \leqslant 1$. If $\{x, y\} \cap D \neq \emptyset$, then $\partial(T) \leqslant \partial(\{x, y\}) + \sum_{u\in N(v)-\{x,y\}} \partial(T_u) \leqslant 1 + \Delta(T) - 2 = \Delta(T) - 1$. If $\{x, y\} \cap D = \emptyset$, then since at most once subtree T_u for $u \in N(v) - \{x, y\}$ can contribute two to the differential of T, it follows that $\partial(T) \leqslant 2 + \Delta(T) - 3 = \Delta(T) - 1$. □

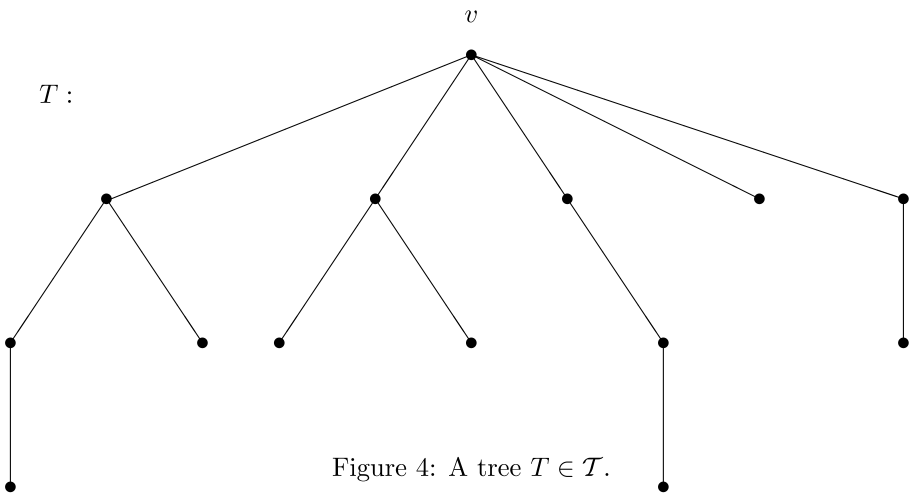

Figure 4: A tree $T \in \mathcal{T}$.

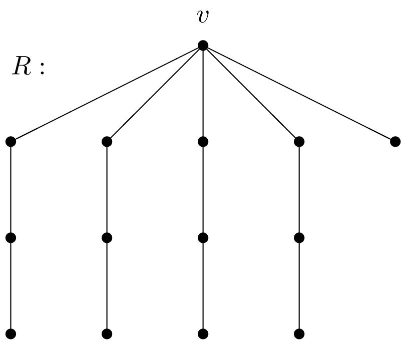

Figure 5: A tree $R \in \mathcal{T}$.

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