TRICYCLIC STEINER TRIPLE SYSTEMS WITH 1-ROTATIONAL SUBSYSTEMS

Robert B. Gardner* and Quan D. Tran

Department of Mathematics

East Tennessee State University

Johnson City, TN 37614

and

Rebecca C. Zijlstra

Department of Mathematics and Statistics

Middle Tennessee State University

Murfreesboro, Tennessee 37312

*Corresponding Author.

Abstract. A Steiner triple system of order v, denoted $STS(v)$, is said to be *tricyclic* if it admits an automorphism whose disjoint cyclic decomposition consists of three cycles. In this paper we give necessary and sufficient conditions for the existence of a tricyclic $STS(v)$ when one of the cycles is of length one. In this case, the $STS(v)$ will contain a subsystem which admits an automorphism consisting of a fixed point and a single cycle. The subsystem is said to be 1-*rotational*.

Keywords: Steiner triple systems, automorphism, tricyclic, rotational.

1. Introduction

A *Steiner triple system* of order v, denoted $STS(v)$, is a v-element set, X, of points, together with a set β, of unordered triples of elements of X, called *blocks*, such that any two points of X are together in exactly one block of β. It is well known that a $STS(v)$ exists if and only if $v \equiv 1$ or 3 (mod 6). An *automorphism* of a STS is a permutation π of X which fixes β. A permutation π of a v-element set is said to be of *type* $[\pi] = [\pi_1, \pi_2, ..., \pi_v]$ if the disjoint cyclic decomposition of π contains π_i cycles of length i (therefore $\sum_{i=1}^{v} i\pi_i = v$). The *orbit* of a block under an automorphism π is the image of the block under the powers of π. A collection of blocks B is said to be a *collection of base blocks for a STS under the permutation* π if the orbits of the blocks of B produce the STS and exactly one block of B occurs in each orbit.

Several types of automorphisms have been explored in connection with the question "For which orders v does there exist a $STS(v)$ admitting an automorphism of the given type?" A *cyclic* $STS(v)$ is one admitting an automorphism of type $[0, 0, ..., 0, 1]$ and exists if and only if $v \equiv 1$ or 3 (mod 6) and $v \neq 9$ [6]. A k-rotational $STS(v)$ admits an automorphism of type $[1, 0, 0, \ldots, 0, k, 0, \ldots, 0]$. A 1-rotational $STS(v)$ exists if and only if $v \equiv 3$ or 9 (mod 24) and a 2-rotational $STS(v)$ exists if and only if $v \equiv 1$, 3, 7, 9, 15 or 19 (mod 24) [7]. A k-transrotational $STS(v)$ admits an automorphism of type $[1, 1, 0, 0, \ldots, 0, k, 0, 0, 0]$ and with $k = 1$ such a system exists if and only if $v \equiv 1$, 7, 9 or 15 (mod 24) [5]. A *bicyclic* $STS(v)$ admits an automorphism of type $[\pi] = [0, 0, \ldots, 0, 1, 0, \ldots, 0, 1, 0, \ldots, 0]$ where $\pi_M = \pi_N = 1$, $M < N$ and $M + N = v$. That is, the disjoint cyclic decomposition of π consists of one cycle of length M and another (larger) cycle of length N. Such a bicyclic $STS(v)$ with $M > 1$ exists if and only if $M \equiv 1$ or 3 (mod 6), $M \neq 9$, $M \mid N$, and $v = M + N \equiv 1$ or 3 (mod 6) [1, 2, 4]. Notice that a 1-rotational STS is a special case of a bicyclic STS, but the existence of 1-rotational STSs does not fit the same pattern as bicyclic STSs (i.e., we cannot simply take $M = 1$ in the previously stated conditions for bicyclic STSs to get the conditions for 1-rotational STSs). The primary purpose of this paper is to give necessary and sufficient conditions for the existence of a STS admitting an automorphism consisting of 3 cycles, the smallest of which is a cycle of length 1 (i.e., a fixed point).

2. Some Tricyclic Steiner Triple Systems

We define a *tricyclic* $STS(v)$ to be one that admits an automorphism either of type $[0,\ldots,0,3,0,\ldots,0]$, $[0,\ldots,0,1,0,\ldots0,2,0,\ldots,0]$, or of type $[0,\ldots,0,1,0,\ldots,0,1,0,\ldots,0,1,0,\ldots,0]$. From the existence of a cyclic $STS(v)$ we readily have:

Theorem 2.1 *A tricyclic $STS(v)$ admitting an automorphism of type $[0,\ldots,0,3,0,\ldots,0]$ exists if and only if $v \equiv 3 \pmod 6$.*

Proof. Of course the condition $v \equiv 3 \pmod 6$ is necessary. For all such v, except $v = 9$, there is a cyclic $STS(v)$. Simply by cubing the cyclic automorphism, we see that the systems are also tricyclic. For $v = 9$, consider the collection of blocks: $(0_0, 0_1, 2_0)$, $(0_0, 0_2, 2_2)$, $(0_0, 1_2, 2_1)$, and $(0_1, 1_1, 1_2)$. This is a collection of base blocks for a tricyclic $STS(9)$ under the automorphism $\pi = (0_0, 1_0, 2_0)(0_1, 1_1, 2_1)(0_2, 1_2, 2_2)$ where the point set is $\mathbb{Z}_3 \times \mathbb{Z}_3$. Here, and throughout, we represent the ordered pair (x, y) as the subscripted pair x_y. ∎

Similarly, we can establish the existence of a large class of tricyclic STSs from the existence of the bicyclic STSs.

Theorem 2.2 *A tricyclic $STS(v)$ admitting an automorphism of type $[\pi] = [0,\ldots,0,1,0,\ldots,0,2,0,\ldots,0]$ where $\pi_M = 1$, $\pi_N = 2$ and $M > 1$ exists if and only if $M \equiv 1$ or $3 \pmod 6$, $M \neq 9$, $M \mid N$ and $v = N + 2M \equiv 1$ or $3 \pmod 6$.*

Proof. First, suppose there is such a system with the point set $\mathbb{Z}_M \bigcup \mathbb{Z}_N \times \mathbb{Z}_2$ admitting the automorphism $\pi = (0, 1, \ldots, M-1)(0_0, 1_0, \ldots, (N-1)_0)(0_1, 1_1, \ldots, (N-1)_1)$. It is rather easy to see that the fixed points of an automorphism form a subsystem of a STS (i.e., if two points of a triple are fixed by the automorphism, then the third point of the triple must also be fixed by the automorphism). By considering π^M we see, therefore, that such a $STS(v)$ has a cyclic subsystem of order M. Therefore, $M \equiv 1$ or $3 \pmod 6$ and $M \neq 9$ is necessary. Also, such a STS must contain some block of the form (x, y_i, z_j) where $x \in \mathbb{Z}_M$ and $y_i, z_j \in \mathbb{Z}_N \times \mathbb{Z}_2$. By applying π^N to this block, we see that $(\pi^N(x), y_i, z_j)$ must also be a block of the STS and therefore $\pi^N(x) = x$ and $M \mid N$ is necessary.

To establish sufficiency, suppose M and N satisfy the stated conditions. Then there is a bicyclic $STS(v)$ admitting an automorphism consisting of a cycle of length M and a cycle of length $2N$. By considering the square of this automorphism, we see that the bicyclic $STS(v)$ is also tricyclic and admits an automorphism of the desired type. ∎

Notice that 2-rotational and 1-transrotational STSs are also examples of tricyclic STSs.

3. Tricyclic Steiner Triple Systems with 1-Rotational Subsystems

We now turn our attention to STSs admitting automorphisms of type $[\pi] = [1, 0, \ldots, 0, 1, 0, \ldots, 0, 1, 0, \ldots, 0]$ where $\pi_1 = \pi_M = \pi_N = 1$, $v = M + N + 1$, and $M < N$. In our discussion, we will let the point set of such a system be $\{\infty\} \bigcup \mathbb{Z}_M \times \{0\} \bigcup \mathbb{Z}_N \times \{1\}$ and let the automorphism be $\pi = (\infty)(0_0, 1_0, \ldots, (M-1)_0)(0_1, 1_1, \ldots, (N-1)_1)$. As in the proof of Theorem 2.2, by considering π^M, we see that the $STS(v)$ contains a 1-rotational subsystem of order $M+1$. Therefore we have:

Lemma 3.1 *If a tricyclic $STS(v)$ exists admitting an automorphism of the type $[\pi] = [1, 0, \ldots, 0, 1, 0, \ldots, 0, 1, 0, \ldots, 0]$ where $\pi_1 = \pi_M = \pi_N = 1$ then $M \equiv 2$ or $8 \pmod{24}$.*

Also, in such a STS there must be some block of the form (x_0, y_1, z_1). By considering the image of this block under π^N, as in Theorem 2.2, we have:

Lemma 3.2 *If a tricyclic $STS(v)$ as described in Lemma 3.1 exists, then $M \mid N$.*

With a pair of points of the form (x_1, y_1) we associate a *pure difference of type* 1 of $\min\{(x -$

$y)(\mathrm{mod}\ N), (y-x)(\mathrm{mod}\ N)\}$. With a pair of points of the form (x_0, y_1) we associate the *mixed difference* $(y-x)$ (mod M). The set of pure differences of type 1 is $\{1, 2, \ldots, N/2\}$ and the set of mixed differences is $\{0, 1, \ldots, M-1\}$. A collection of base blocks for the desired type of STS must contain a block of the form $(\infty, x_1, (x+N/2)_1)$. Notice that this block contains a pair of points with the associated pure difference of type 1 of $N/2$. Therefore, constructing the desired type of STS is equivalent to presenting a collection of blocks on the point set $\mathbb{Z}_M \times \{0\} \bigcup \mathbb{Z}_N \times \{1\}$ such that the differences associated with the pairs of points of these blocks precisely cover the set of pure differences of type 1 of $\{1, 2, \ldots, N/2-1\}$ and the set of mixed differences of $\{0, 1, \ldots, M-1\}$. Such a collection of blocks along with a collection of base blocks for a 1-rotational $STS(M+1)$ on the point set $\{\infty\} \bigcup \mathbb{Z}_M \times \{1\}$ (under the obvious automorphism) and the block $(\infty, 0_1, (N/2)_1)$ form a collection of base blocks for a tricyclic $STS(1+M+N)$ with a 1-rotational subsystem under π.

We have a final necessary condition:

Lemma 3.3 *If a tricyclic $STS(v)$ as described in Lemma 3.1 exists, then $N = kM$ where $k \equiv 2, 3, 6$ or 11 (mod 12) whenever $M \equiv 2$ (mod 24). If $M \equiv 8$ (mod 24), then $k \equiv 0$ or 2 (mod 3).*

Proof. A base block of the form (x_0, y_1, z_1) covers two mixed differences and one pure difference of type 1. One of the mixed differences must be congruent to the sum of the other two differences modulo M. Since M is even, either zero or two of these differences is/are odd. If $3 \mid N$, then a possible base block is one of the form $(x_1, (x+N/3)_1, (x+2N/3)_1)$. A block of this type is said to be a *short orbit block* since the length of its orbit under π is precisely one-third the length of the orbit of any other block on the points $\mathbb{Z}_N \times \{1\}$. A short orbit block covers the pure difference of type 1 of $N/3$ only, and $N/3$ is even. A base block of the form (x_1, y_1, z_1) (other than a short orbit block) covers three distinct pure differences of type 1. These three differences satisfy either the condition that one is the sum of the other two, or the condition that all three sum to 0 modulo N. In either case, either zero or two of these differences is/are odd. So, a collection of blocks of the form (x_0, y_1, z_1) or (x_1, y_1, z_1) covers an even number of odd differences. Therefore, the number of odd differences in the set $\{0, 1, \ldots, M-1\} \bigcup \{1, 2, \ldots, N/2 - 1\}$ must be even. From this, the lemma follows. ∎

We now show that the necessary conditions of Lemmas 3.1–3.3 are sufficient in a series of constructions.

Lemma 3.4 *If $M \equiv 2 \pmod{24}$ and $k \equiv 2, 3, 6$ or $11 \pmod{12}$, then there exists a tricyclic $STS(v)$ as described above.*

Proof. Consider the given collections of blocks.

Case 1. Suppose that $M \equiv 2$ (mod 24) and $k \equiv 2$ (mod 12).

If $M = 26$ and $k = 2$, consider the following collection of blocks:

$(0_1, 7_1, 18_1)$, $(0_1, 8_1, 17_1)$, $(0_1, 13_1, 25_1)$, $(0_1, 14_1, 24_1)$, $(\infty, 0_1, 26_1)$,

$(0_0, 0_1, 15_1)$, $(0_0, 1_1, 17_1)$, $(0_0, 8_1, 28_1)$, $(0_0, 7_1, 29_1)$, $(0_0, 11_1, 30_1)$, $(0_0, 10_1, 31_1)$,

$(0_0, 9_1, 32_1)$, $(0_0, 12_1, 18_1)$, $(0_0, 13_1, 16_1)$, $(0_0, 14_1, 19_1)$, $(0_0, 20_1, 24_1)$, $(0_0, 23_1, 25_1)$, $(0_0, 21_1, 22_1)$.

Otherwise, consider the following collection of blocks:

$\left(0_1, \left(\frac{(k-1)M+10}{6} - 2r\right)_1, \left(\frac{(k-1)M}{2} - r\right)_1\right)$ for $r = 1, 2, \ldots, \frac{(k-1)M-2}{12}$,

$\left(0_1, \left(\frac{(k-1)M-8}{6} - 2r\right)_1, \left(\frac{(k-1)M-5}{3} - r\right)_1\right)$ for $r = 1, 2, \ldots, \frac{(k-1)M-50}{24}$ (omit if $M = 26$),

$\left(0_1, \left(\frac{(k-1)M-14}{12} - 2r\right)_1, \left(\frac{7(k-1)M-14}{24} - r\right)_1\right)$ for $r = 1, 2, \ldots, \frac{(k-1)M-50}{24}$,

$\left(0_1, \left(\frac{(k-1)M+10}{12}\right)_1, \left(\frac{7(k-1)M-14}{24}\right)_1\right)$, $\left(0_1, 1_1, \left(\frac{5(k-1)M+14}{24}\right)_1\right)$,

$$\left(0_1, \left(\frac{(k-1)M-14}{12}\right)_1, \left(\frac{(k-1)M-5}{3}\right)_1\right) \text{ (omit if } M=26\text{)},$$

$$\left(0_1, \left(\frac{(k-1)M-8}{6}\right)_1, \left(\frac{5(k-1)M-10}{12}\right)_1\right), \left(\infty, 0_1, \left(\frac{kM}{2}\right)_1\right),$$

$$\left(0_0, (M-r)_1, \left(\frac{(k+1)M-2}{2}+r\right)_1\right) \text{ for } r=1,2,\ldots,\frac{M-2}{4},$$

$$\left(0_0, \left(\frac{M-2}{2}-r\right)_1, \left(\frac{kM-2}{2}+r\right)_1\right) \text{ for } r=1,2,\ldots,\frac{M-2}{8},$$

$$\left(0_0, \left(\frac{3M-6}{8}-r\right)_1, \left(\frac{(4k+1)M+6}{8}+r\right)_1\right) \text{ for } r=1,2,\ldots,\frac{M-18}{8},$$

$$\left(0_0, \left(\frac{M+6}{8}\right)_1, \left(\frac{(4k+1)M-2}{8}\right)_1\right), \left(0_0, \left(\frac{3M-2}{4}\right)_1, \left(\frac{(2k+1)M-2}{4}\right)_1\right), \text{ and}$$

$$\left(0_0, \left(\frac{M-2}{2}\right)_1, \left(\frac{(2k+1)M+2}{4}\right)_1\right).$$

Case 2. Suppose that $M \equiv 2 \pmod{24}$ and $k \equiv 3 \pmod{12}$. Consider the following collection of blocks:

$$\left(0_1, \left(\frac{kM}{3}\right)_1, \left(\frac{2kM}{3}\right)_1\right), \left(\infty, 0_1, \left(\frac{kM}{2}\right)_1\right),$$

$$\left(0_1, \left(\frac{kM}{6}-2r\right)_1, \left(\frac{kM}{2}-r\right)_1\right) \text{ for } r=1,2,\ldots,\frac{(k-1)M-4}{12},$$

$$\left(0_1, \left(\frac{kM+6}{6}-2r\right)_1, \left(\frac{kM}{3}-r\right)_1\right) \text{ for } r=1,2,\ldots,\frac{(k-1)M-4}{12},$$

$$\left(0_0, \left(\frac{M+4}{3}-r\right)_1, \left(\frac{(3k+3)M-12}{12}+r\right)_1\right) \text{ for } r=1,2,\ldots,\frac{M+10}{12},$$

$$\left(0_0, \left(\frac{M+1}{3}+r\right)_1, \left(\frac{(3k+5)M+8}{12}-r\right)_1\right) \text{ for } r=1,2,\ldots,\frac{M-2}{12},$$

$$\left(0_0, \left(\frac{(3k+5)M-4}{12}+r\right)_1, \left(\frac{(8k+6)M+12}{12}-r\right)_1\right) \text{ for } r=1,2,\ldots,\frac{M-2}{12},$$

$$\left(0_0, \left(\frac{(4k+3)M}{6}+r\right)_1, \left(\frac{(13k+7)M+4}{12}-r\right)_1\right) \text{ for } r=1,2,\ldots,\frac{M-2}{24},$$

$$\left(0_0, \left(\frac{(16k+13)M-2}{24}+r\right)_1, \left(\frac{(16k+17)M+14}{24}-r\right)_1\right) \text{ for } r=1,2,\ldots,\frac{M-2}{12},$$

$$\left(0_0, \left(\frac{(16k+17)M-10}{24}+r\right)_1, \left(\frac{(16k+21)M+30}{24}-r\right)_1\right) \text{ for } r=1,2,\ldots,\frac{M-2}{24},$$

$$\left(0_0, \left(\frac{(16k+18)M-12}{24}+r\right)_1, \left(\frac{(26k+18)M}{24}-r\right)_1\right) \text{ for } r=1,2,\ldots,\frac{M-2}{24},$$

$$\left(0_0, \left(\frac{(26k+15)M+6}{24}+r\right)_1, \left(\frac{(26k+17)M+2}{24}-r\right)_1\right) \text{ for } r=1,2,\ldots,\frac{M-26}{24},$$

$$\left(0_0, \left(\frac{(13k+7)M+4}{12}\right)_1, \left(\frac{(26k+16)M+4}{24}\right)_1\right).$$

Case 3. Suppose that $M \equiv 2 \pmod{24}$ and $k \equiv 6 \pmod{12}$. Consider the following collection of blocks:

$$\left(0_1, \left(\frac{kM}{4}\right)_1, \left(\frac{5kM}{12}\right)_1\right), \left(0_1, \left(\frac{kM}{3}\right)_1, \left(\frac{2kM}{3}\right)_1\right),$$

$$\left(0_1, \left(\frac{(k+1)M-8}{6}+r\right)_1, \left(\frac{(2k-1)M+8}{6}-r\right)_1\right) \text{ for } r=1,2,\ldots,\frac{(k-2)M+4}{12},$$

$$\left(0_1, \left(\frac{kM}{3}+r\right)_1, \left(\frac{kM-2}{2}-r\right)_1\right) \text{ for } r=1,2,\ldots,\frac{kM-24}{12},$$

$$\left(\infty, 0_1, \left(\frac{kM}{2}\right)_1\right), \left(0_0, 0_1, \left(\frac{5kM}{12}-1\right)_1\right),$$

$$\left(0_0, \left(\frac{M-2}{2}+r\right)_1, \left(\frac{(k+3)M-6}{6}-r\right)_1\right) \text{ for } r=1,2,\ldots,\frac{M-8}{6},$$

$$\left(0_0, \left(\frac{M-5}{3}\right)_1, \left(\frac{M-2}{3}\right)_1\right), \left(0_0, \left(\frac{M-6}{4}\right)_1, \left(\frac{(4k+1)M-2}{12}\right)_1\right),$$

$$\left(0_0, \left(\frac{M-6}{4}-r\right)_1, \left(\frac{(2k+3)M-18}{12}+r\right)_1\right) \text{ for } r=1,2,\ldots,\frac{M-14}{12},$$

$$\left(0_0, \left(\frac{M-2}{12}+r\right)_1, \left(\frac{(4k+1)M-2}{12}-r\right)_1\right) \text{ for } r=1,2,\ldots,\frac{M-14}{12},$$

$$\left(0_0, \left(\frac{9M-18}{12}-r\right)_1, \left(\frac{(2k+9)M-30}{12}+r\right)_1\right) \text{ for } r=1,2,\ldots,\frac{M-2}{12},$$

$$\left(0_0, \left(\frac{11M-46}{12}+r\right)_1, \left(\frac{(4k+11)M-34}{12}-r\right)_1\right) \text{ for } r=1,2,\ldots,\frac{M-14}{12},$$

$$\left(0_0, (M-3)_1, \left(\frac{(k+6)M-24}{6}\right)_1\right), \text{ and } \left(0_0, (M-1)_1, \left(\frac{(k+2)M-4}{2}\right)_1\right).$$

Case 4. Suppose that $M \equiv 2 \pmod{24}$ and $k \equiv 11 \pmod{12}$. Consider the following collection of blocks:

$$\left(0_1, \left(\frac{(k-1)M-8}{6}-2r\right)_1, \left(\frac{(k-1)M-2}{2}-r\right)_1\right) \text{ for } r=1,2,\ldots,\frac{(k-1)M-20}{12},$$

$$\left(0_1, \left(\frac{(k-1)M-2}{6}-2r\right)_1, \left(\frac{(k-1)M-2}{3}-r\right)_1\right) \text{ for } r=1,2,\ldots,\frac{(k-1)M-44}{24},$$

$$\left(0_1, \left(\frac{(k-1)M-8}{12}-2r\right)_1, \left(\frac{7(k-1)M+4}{24}-r\right)_1\right) \text{ for } r=1,2,\ldots,\frac{(k-1)M-44}{24},$$

$$\left(0_1, \left(\frac{(k-1)M+16}{12}\right)_1, \left(\frac{7(k-1)M+4}{24}\right)_1\right), \left(0_1, 1_1, \left(\frac{5(k-1)M+20}{24}\right)_1\right),$$

$$\left(0_1, \left(\frac{(k-1)M-8}{12}\right)_1, \left(\frac{(k-1)M-2}{3}\right)_1\right), \left(0_1, \left(\frac{(k-1)M-8}{6}\right)_1, \left(\frac{5(k-1)M-4}{12}\right)_1\right),$$

$$\left(0_1, \left(\frac{(k-1)M-2}{6}\right)_1, \left(\frac{(k-1)M}{2}\right)_1\right), \left(\infty, 0_1, \left(\frac{kM}{2}\right)_1\right),$$

$$\left(0_0, 0_1, \left(\frac{(k-1)M-2}{2}\right)_1\right), \left(0_0, \left(\frac{M-10}{8}\right)_1, \left(\frac{4k-1)M-6}{8}\right)_1\right),$$

$$\left(0_0, \left(\frac{M-2}{8}\right)_1, \left(\frac{(4k-3)M+6}{8}\right)_1\right), \left(0_0, \left(\frac{M-2}{4}\right)_1, \left(\frac{kM-4}{2}\right)_1\right),$$

$$\left(0_0, \left(\frac{M+2}{4}\right)_1, \left(\frac{(2k+1)M-6}{4}\right)_1\right),$$

$$\left(0_0, \left(\frac{M-4}{2}+r\right)_1, \left(\frac{(k+1)M-2}{2}-r\right)_1\right) \text{ for } r=1,2,\ldots,\frac{M-2}{4},$$

$$\left(0_0, r_1, \left(\frac{kM-4}{2}-r\right)_1\right) \text{ for } r=1,2,\ldots,\frac{M-18}{8}, \text{ and}$$

$$\left(0_0, \left(\frac{M+6}{8}+r\right)_1, \left(\frac{(4k-1)M-6}{8}-r\right)_1\right) \text{ for } r=1,2,\ldots,\frac{M-18}{8}.$$

In each case, the given collection of blocks, along with a collection of base blocks for a 1-rotational $STS(M+1)$ on the point set $\{\infty\}\bigcup \mathbb{Z}_M \times \{0\}$ under the automorphism $(\infty)(0_0, 1_0, \ldots, (M-1)_0)$, forms a collection of base blocks for a STS of the desired type. ∎

Lemma 3.5 *If $M \equiv 8$ (mod 24) and $k \equiv 0$ or 2 (mod 3), then there exists a tricyclic $STS(v)$ as described above.*

Proof. Consider the given collections of blocks.

Case 1. Suppose that $M \equiv 8$ (mod 24) and $k \equiv 0$ (mod 12). Consider the following collection of blocks:

$$\left(0_1, \left(\frac{kM}{6}+r\right)_1, \left(\frac{kM}{3}-r\right)_1\right) \text{ for } r = 1, 2, \ldots, \frac{kM-12}{12},$$

$$\left(0_1, \left(\frac{(2k+1)M-8}{6}+r\right)_1, \left(\frac{(3k-1)M+2}{6}-r\right)_1\right) \text{ for } r = 1, 2, \ldots, \frac{(k-2)M-8}{12},$$

$$\left(0_1, \left(\frac{kM}{4}\right)_1, \left(\frac{5kM}{12}\right)_1\right), \left(0_1, \left(\frac{kM}{3}\right)_1, \left(\frac{2kM}{3}\right)_1\right), \left(\infty, 0_1, \left(\frac{kM}{2}\right)_1\right),$$

$$\left(0_0, \left(\frac{M-8}{6}\right)_1, \left(\frac{M-2}{6}\right)_1\right), \left(0_0, \left(\frac{7M-20}{12}\right)_1, \left(\frac{(6k+5)M-16}{12}\right)_1\right),$$

$$\left(0_0, \left(\frac{M-2}{6}+r\right)_1, \left(\frac{(k+1)M-8}{6}-r\right)_1\right) \text{ for } r = 1, 2, \ldots, \frac{M-8}{6},$$

$$\left(0_0, \left(\frac{5M-16}{12}-r\right)_1, \left(\frac{(4k+5)M-16}{12}+r\right)_1\right) \text{ for } r = 1, 2, \ldots, \frac{M-8}{12},$$

$$\left(0_0, \left(\frac{7M-20}{12}+r\right)_1, \left(\frac{(6k+7)M-20}{12}-r\right)_1\right) \text{ for } r = 1, 2, \ldots, \frac{M-8}{12},$$

$$\left(0_0, \left(\frac{8M-28}{12}+r\right)_1, \left(\frac{(4k+10)M-32}{12}-r\right)_1\right) \text{ for } r = 1, 2, \ldots, \frac{M-8}{12},$$

$$\left(0_0, \left(\frac{11M-40}{12}+r\right)_1, \left(\frac{(6k+11)M-52}{12}-r\right)_1\right) \text{ for } r = 1, 2, \ldots, \frac{M-20}{12},$$

$\left(0_0, \left(\frac{11M-40}{12}\right)_1, \left(\frac{(2k+11)M-52}{12}\right)_1\right)$, $\left(0_0, (M-3)_1, \left(\frac{(k+2)M-8}{2}\right)_1\right)$, and

$\left(0_0, (M-1)_1, \left(\frac{(5k+12)M-24}{12}\right)_1\right)$.

<u>Case 2.</u> Suppose that $M \equiv 8 \pmod{24}$ and $k \equiv 2 \pmod 6$. Consider the following collection of blocks:

$\left(0_1, \left(\frac{(k-1)M+4}{6} - 2r\right)_1, \left(\frac{(k-1)M}{2} - r\right)_1\right)$ for $r = 1, 2, \ldots, \frac{(k-1)M-8}{12}$,

$\left(0_1, \left(\frac{(k-1)M-2}{6} - 2r\right)_1, \left(\frac{(k-1)M-2}{3} - r\right)_1\right)$ for $r = 1, 2, \ldots, \frac{(k-1)M-8}{24}$,

$\left(0_1, \left(\frac{(k-1)M-20}{12} - 2r\right)_1, \left(\frac{7(k-1)M-8}{24} - r\right)_1\right)$ for $r = 1, 2, \ldots, \frac{(k-1)M-56}{24}$ (omit if $M = 8$),

$\left(0_1, 1_1, \left(\frac{5(k-1)M+32}{24}\right)_1\right)$, $\left(0_1, \left(\frac{(k-1)M-2}{6}\right)_1, \left(\frac{5(k-1)M-4}{12}\right)_1\right)$,

$\left(0_1, \left(\frac{(k-1)M-20}{12}\right)_1, \left(\frac{(k-1)M-2}{3}\right)_1\right)$, $\left(\infty, 0_1, \left(\frac{kM}{2}\right)_1\right)$,

$\left(0_0, (M-r)_1, \left(\frac{(k+1)M-4}{2} + r\right)_1\right)$ for $r = 1, 2, \ldots, \frac{M}{4}$,

$\left(0_0, \left(\frac{M-4}{2} - r\right)_1, \left(\frac{kM-2}{2} + r\right)_1\right)$ for $r = 1, 2, \ldots, \frac{M-16}{8}$,

$\left(0_0, \left(\frac{3M-8}{8} - r\right)_1, \left(\frac{(4k+1)M}{8} + r\right)_1\right)$ for $r = 1, 2, \ldots, \frac{M-16}{8}$,

$\left(0_0, \left(\frac{M}{8}\right)_1, \left(\frac{(4k+1)M-8}{8}\right)_1\right)$, $\left(0_0, \left(\frac{M-4}{2}\right)_1, \left(\frac{(2k+1)M-4}{4}\right)_1\right)$ (omit if $M = 8$),

$\left(0_0, \left(\frac{3M-4}{4}\right)_1, \left(\frac{(2k+1)M}{4}\right)_1\right)$, and $\left(0_0, \left(\frac{3M-8}{8}\right)_1, \left(\frac{(4k+1)M-16}{8}\right)_1\right)$.

<u>Case 3.</u> Suppose that $M \equiv 8 \pmod{24}$ and $k \equiv 3 \pmod{12}$. Consider the following collection of blocks:

$$\left(0_1, \left(\frac{kM}{3}\right)_1, \left(\frac{2kM}{3}\right)_1\right), \left(\infty, 0_1, \left(\frac{kM}{2}\right)_1\right),$$

$$\left(0_1, \left(\frac{kM}{6} - 2r\right)_1, \left(\frac{kM}{2} - r\right)_1\right) \text{ for } r = 1, 2, \ldots, \frac{(k-1)M-4}{12},$$

$$\left(0_1, \left(\frac{kM+6}{6} - 2r\right)_1, \left(\frac{kM}{3} - r\right)_1\right) \text{ for } r = 1, 2, \ldots, \frac{(k-1)M-4}{12},$$

$$\left(0_1, \left(\frac{M+1}{3} - r\right)_1, \left(\frac{(3k+3)M-12}{12} + r\right)_1\right) \text{ for } r = 1, 2, \ldots, \frac{M+4}{12},$$

$$\left(0_0, \left(\frac{M-8}{12} + r\right)_1, \left(\frac{(5k+2)M-4}{12} - r\right)_1\right) \text{ for } r = 1, 2, \ldots, \frac{M-8}{12},$$

$$\left(0_0, \left(\frac{(5k+3)M+12}{12} - r\right)_1, \left(\frac{(8k+2)M-16}{12} + r\right)_1\right) \text{ for } r = 0, 1, 2, \ldots, \frac{M+4}{12},$$

$$\left(0_0, \left(\frac{(5k+2)M-4}{12}\right)_1, \left(\frac{(10k+1)M+4}{12}\right)_1\right)$$

$$\left(0_0, \left(\frac{(5k+3)M}{12} + r\right)_1, \left(\frac{(10k+4)M+16}{12} - r\right)_1\right) \text{ for } r = 1, 2, \ldots, \frac{M-8}{12},$$

$$\left(0_0, \left(\frac{(10k+4)M+16}{12}\right)_1, \left(\frac{(10k+2)M+32}{12}\right)\right), \left(0_0, \left(\frac{(10k+3)M+12}{12}\right)_1, \left(\frac{(10k+5)M+8}{12}\right)_1\right),$$

$$\left(0_0, \left(\frac{(10k+4)M+16}{12} + r\right)_1, \left(\frac{(10k+6)M}{12} - r\right)_1\right) \text{ for } r = 1, 2, \ldots, \frac{M-20}{12}.$$

If $M = 32$, also take the two blocks:

$$\left(0_0, \left(\frac{(10k+1)M+16}{12} + r\right)_1, \left(\frac{(10k+1)M+52}{12} - r\right)_1\right) \text{ for } r = 0, 1.$$

If $M > 32$, instead of the last two blocks, take the blocks:

$$\left(0_0, \left(\frac{(10k+3)M+12}{12} - r\right)_1, \left(\frac{(10k+1)M+16}{12} + r\right)_1\right) \text{ for } r = 1, 2, \ldots, \frac{M-32}{24},$$

$$\left(0_0, \left(\frac{(20k+5)M+56}{24} - r\right)_1, \left(\frac{(20k+3)M+48}{24} + r\right)_1\right) \text{ for } r = 1, 2, \ldots, \frac{M-32}{24},$$

$$\left(0_0, \left(\frac{(10k+1)M+16}{12}\right)_1, \left(\frac{(10k+2)M+20}{12}\right)_1\right), \left(0_0, \left(\frac{(20k+3)M+24}{24}\right)_1, \left(\frac{(20k+3)M+48}{24}\right)_1\right).$$

Case 4. Suppose that $M \equiv 8 \pmod{24}$ and $k \equiv 5 \pmod{12}$. Consider the following collection of blocks:

$$\left(0_1, \left(\frac{(k-1)M+10}{6}+r\right)_1, \left(\frac{(k-1)M+7}{3}-r\right)_1\right) \text{ for } r = 1, 2, \ldots, \frac{(k-1)M-8}{24},$$

$$\left(0_1, \left(\frac{5(k-1)M+56}{24}+r\right)_1, \left(\frac{7(k-1)M+64}{24}-r\right)_1\right) \text{ for } r = 1, 2, \ldots, \frac{(k-1)M-32}{24},$$

$$\left(0_1, \left(\frac{5(k-1)M+56}{24}\right)_1, \left(\frac{3(k-1)M+24}{8}\right)_1\right), \left(0_1, \left(\frac{(k-1)M+8}{4}\right)_1, \left(\frac{5(k-1)M+44}{12}\right)_1\right),$$

$$\left(0_1, \left(\frac{(k-1)M+12}{4}\right)_1, \left(\frac{5(k-1)M+32}{12}\right)_1\right), \left(0_1, \left(\frac{(k-1)M+7}{3}\right)_1, \left(\frac{5(k-1)M+20}{12}\right)_1\right),$$

$$\left(0_1, \left(\frac{(k-1)M+7}{3}+r\right)_1, \left(\frac{(k-1)M+4}{2}-r\right)_1\right) \text{ for } r = 1, 2, \ldots, \frac{(k-1)M-8}{24},$$

$$\left(0_1, \left(\frac{3(k-1)M+24}{8}+r\right)_1, \left(\frac{11(k-1)M+56}{24}-r\right)_1\right) \text{ for } r = 1, 2, \ldots, \frac{(k-1)M-56}{24},$$

$$\left(0_0, \left(\frac{M-2}{2}+r\right)_1, \left(\frac{(k+1)M-6}{2}-r\right)_1\right) \text{ for } r = 1, 2, \ldots, \frac{M-8}{4},$$

$$\left(0_0, (r-1)_1, \left(\frac{kM-4}{2}-r\right)_1\right) \text{ for } r = 1, 2, \ldots, \frac{M-16}{8},$$

$$\left(0_0, \left(\frac{M}{8}+r\right)_1, \left(\frac{(4k-1)M-8}{8}-r\right)_1\right) \text{ for } r = 1, 2, \ldots, \frac{M-16}{8},$$

$$\left(0_0, \left(\frac{M-16}{8}\right)_1, \left(\frac{(4k-1)M-8}{8}\right)\right)_1, \left(0_0, \left(\frac{M-8}{8}\right)_1, \left(\frac{M}{8}\right)_1\right),$$

$$\left(0_0, \left(\frac{M-4}{4}\right)_1, \left(\frac{kM-4}{2}\right)_1\right), \left(0_0, \left(\frac{M}{4}\right)_1, \left(\frac{(2k+1)M-8}{4}\right)_1\right),$$

$$\left(0_0, \left(\frac{M-2}{2}\right)_1, \left(\frac{(k+1)M-4}{2}\right)_1\right), (0_0, (M-3)_1, (M-1)_1), \text{ and } \left(\infty, 0_1, \left(\frac{kM}{2}\right)_1\right).$$

Case 5. Suppose that $M \equiv 8 \pmod{24}$ and $k \equiv 6 \pmod{12}$. Consider the following collection of blocks:

$$\left(0_1, \left(\frac{kM}{3}\right)_1, \left(\frac{2kM}{3}\right)_1\right), \left(0_1, \left(\frac{kM-6}{6}\right)_1, \left(\frac{kM-3}{3}\right)_1\right), \left(\infty, 0_1, \left(\frac{kM}{2}\right)_1\right),$$

$$\left(0_1, \left(\frac{(k+1)M-2}{6}+r\right)_1, \left(\frac{kM-3}{3}-r\right)_1\right) \text{ for } r = 1, 2, \dots \frac{(k-1)M-16}{12},$$

$$\left(0_1, \left(\frac{kM}{3}+r\right)_1, \left(\frac{(3k-1)M+2}{6}-r\right)_1\right) \text{ for } r = 1, 2, \dots, \frac{(k-1)M-4}{12}.$$

If $M = 8$, also take these four blocks:

$$\left(0_0, 1_1, \left(\frac{(3k+1)M+4}{12}\right)_1\right), \left(0_0, \left(\frac{(3k+1)M-8}{12}\right)_1, \left(\frac{(9k+1)M-20}{12}\right)_1\right),$$

$$\left(0_0, \left(\frac{(9k+1)M-20}{12}\right)_1, \left(\frac{(11k+1)M-44}{12}\right)_1\right), \left(0_0, \left(\frac{(3k+1)M+16}{12}\right)_1, \left(\frac{(5k+1)M+28}{12}\right)_1\right).$$

If $M = 32$, instead of the last four blocks, take these blocks:

$$\left(0_0, \left(\frac{-kM+4}{4}\right)_1, \left(\frac{M+4}{12}\right)_1\right),$$

$$\left(0_0, \left(\frac{M+10}{6}-r\right)_1, \left(\frac{kM-6}{6}+r\right)_1\right) \text{ for } r = 1, 2, \dots, \frac{M+4}{12},$$

$$\left(0_0, \left(\frac{M+7}{3}-r\right)_1, \left(\frac{(3k+1)M+4}{6}+r\right)_1\right) \text{ for } r = 1, 2, \dots, \frac{M+4}{12},$$

$$\left(0_0, \left(\frac{(-3k+1)M+16}{12}-r\right)_1, \left(\frac{(3k-1)M+8}{12}+r\right)_1\right) \text{ for } r = 1, 2, \dots, \frac{M-8}{12},$$

$$\left(0_0, \left(\frac{9M+48}{24}-r\right)_1, \left(\frac{(4k+9)M+24}{24}+r\right)_1\right) \text{ for } r = 1, 2, \dots, \frac{M-8}{24},$$

$$\left(0_0, \left(\frac{(-3k+1)M+16}{12}\right)_1, \left(\frac{(-k+3)M+12}{12}\right)_1\right), \left(0_0, (M-1)_1, \left(\frac{(k+5)M+8}{6}\right)_1\right),$$

$$\left(0_0, \left(\frac{(-3k+1)M+40}{12}\right)_1, \left(\frac{(-k+3)M+24}{12}\right)_1\right), \left(0_0, (M-3)_1, \left(\frac{(k+5)M-16}{6}\right)_1\right),$$

$$\left(0_0, \left(\frac{(-3k+1)M+28}{12}\right)_1, \left(\frac{(-k+1)M+52}{12}\right)_1\right), \left(0_0, (M-5)_1, \left(\frac{(k+6)M-12}{6}\right)_1\right).$$

If $M > 32$, instead of the last two collections of blocks, take these blocks:

$$\left(0_0, \left(\frac{-kM+4}{4}\right)_1, \left(\frac{M+4}{12}\right)_1\right),$$

$$\left(0_0, \left(\frac{M+10}{6} - r\right)_1, \left(\frac{kM-6}{6} + r\right)_1\right) \text{ for } r = 1, 2, \ldots, \frac{M+4}{12},$$

$$\left(0_0, \left(\frac{M+7}{3} - r\right)_1, \left(\frac{(3k+1)M+4}{6} + r\right)_1\right) \text{ for } r = 1, 2, \ldots, \frac{M+4}{12},$$

$$\left(0_0, \left(\frac{(-3k+1)M+16}{12} - r\right)_1, \left(\frac{(3k-1)M+8}{12} + r\right)_1\right) \text{ for } r = 1, 2, \ldots, \frac{M-8}{12},$$

$$\left(0_0, \left(\frac{9M+48}{24} - r\right)_1, \left(\frac{(4k+9)M+24}{24} + r\right)_1\right) \text{ for } r = 1, 2, \ldots, \frac{M-8}{24},$$

$$\left(0_0, \left(\frac{(-3k+1)M+4}{12} + r\right)_1, \left(\frac{(-5k+3)M+24}{12} - r\right)_1\right) \text{ for } r = 1, 2, \ldots, \frac{M+16}{24},$$

$$\left(0_0, \left(\frac{(-6k+3)M+24}{24} + r\right)_1, \left(\frac{(-2k+7)M+64}{24} - r\right)_1\right) \text{ for } r = 1, 2, \ldots, \frac{M+16}{24},$$

$$\left(0_0, \left(\frac{(-6k+4)M+40}{24}\right)_1, \left(\frac{(-2k+8)M+56}{24} - r\right)_1\right) \text{ for } r = 1, 2, \ldots, \frac{M-32}{24},$$

$$\left(0_0, \left(\frac{(-2k+7)M+64}{24}\right)_1, \left(\frac{(2k+9)M+96}{24}\right)_1\right), \left(0_0, \left(\frac{(2k+9)M+72}{24}\right)_1, \left(\frac{(6k+11)M+8}{24}\right)_1\right),$$

$$\left(0_0, \left(\frac{(2k+9)M+48}{24}\right)_1, \left(\frac{(6k+11)M+32}{24}\right)_1\right), \left(0_0, \left(\frac{(-2k+8)M+56}{24}\right)_1, \left(\frac{(2k+8)M+104}{24}\right)_1\right),$$

$$\left(0_0, \left(\frac{(k+4)M+40}{12}\right)_1, \left(\frac{(-k+5)M+20}{12}\right)_1\right),$$

$$\left(0_0, \left(\frac{(6k+11)M+32}{24} + r\right)_1, \left(\frac{(10k+11)M+8}{24} - r\right)_1\right) \text{ for } r = 1, 2, \ldots, \frac{M-56}{24},$$

$$\left(0_0, \left(\frac{(2k+9)M+48}{24} - r\right)_1, \left(\frac{(6k+9)M+96}{24} + r\right)_1\right) \text{ for } r = 1, 2, \ldots, \frac{M-80}{24}.$$

Case 6. Suppose that $M \equiv 8 \pmod{24}$ and $k \equiv 9 \pmod{12}$. Consider the following collection of blocks:

$$\left(0_1, \left(\frac{(k+1)M-8}{6}+r\right)_1, \left(\frac{kM}{3}-r\right)_1\right) \text{ for } r = 1, 2, \ldots, \frac{(k-1)M-4}{12},$$

$$\left(0_1, \left(\frac{(2k+1)M-2}{6}+r\right)_1, \left(\frac{kM}{2}-r\right)_1\right) \text{ for } r = 1, 2, \ldots, \frac{(k-1)M-4}{12},$$

$$\left(0_1, (-1+r)_1, \left(\frac{kM+6}{6}-r\right)_1\right) \text{ for } r = 1, 2, \ldots, \frac{M+4}{12},$$

$$\left(0_0, \left(\frac{M+4}{12}\right)_1, \left(\frac{(3k+2)M-4}{12}\right)_1\right), \left(0_0, \left(\frac{(-4k+5)M+8}{24}\right)_1, \left(\frac{(4k+9)M}{24}\right)_1\right),$$

$$\left(0_0, \left(\frac{M+4}{12}+r\right)_1, \left(\frac{(2k+1)M+16}{12}-r\right)_1\right) \text{ for } r = 1, 2, \ldots, \frac{M+4}{12},$$

$$\left(0_0, \left(\frac{2M+8}{12}+r\right)_1, \left(\frac{(-k+1)M+4}{6}-r\right)_1\right) \text{ for } r = 1, 2, \ldots, \frac{M-8}{12},$$

$$\left(0_0, \left(\frac{3M}{12}+r\right)_1, \left(\frac{(4k+5)M-4}{12}-r\right)_1\right) \text{ for } r = 1, 2, \ldots, \frac{M-8}{24},$$

$$\left(0_0, \left(\frac{7M-8}{24}+r\right)_1, \left(\frac{(-4k+7)M+16}{24}-r\right)_1\right) \text{ for } r = 1, 2, \ldots, \frac{M-8}{12},$$

$$\left(0_0, \left(\frac{(-k+1)M-2}{6}+r\right)_1, \left(\frac{(k+2)M+2}{6}-r\right)_1\right) \text{ for } r = 1, 2, \ldots, \frac{M-8}{24},$$

$$\left(0_0, \left(\frac{(4k+9)M}{24}-r\right)_1, \left(\frac{(12k+9)M}{24}+r\right)_1\right) \text{ for } r = 1, 2, \ldots, \frac{M-8}{24},$$

$$\left(0_0, \left(\frac{(12k+10)M-8}{24}+r\right)_1, \left(\frac{(5k+3)M}{6}-r\right)_1\right) \text{ for } r = 1, 2, \ldots, \frac{M-8}{24}, \text{ and}$$

$$\left(\infty, 0_1, \left(\frac{kM}{2}\right)_1\right).$$

Case 7. Suppose that $M \equiv 8 \pmod{24}$ and $k \equiv 11 \pmod{12}$. Consider the following collection of blocks:

$$\left(0_1, \left(\frac{(k-1)M+4}{6}+r\right)_1, \left(\frac{(2k-2)M+2}{6}-r\right)_1\right) \text{ for } r = 1, 2, \ldots, \frac{(k-1)M-32}{24},$$

$$\left(0_1, \left(\frac{5(k-1)M+32}{24}+r\right)_1, \left(\frac{7(k-1)M+40}{24}-r\right)_1\right) \text{ for } r = 1, 2, \ldots, \frac{(k-1)M-32}{24},$$

$$\left(0_1, \left(\frac{8(k-1)M+32}{24}+r\right)_1, \left(\frac{12(k-1)M}{24}-r\right)_1\right) \text{ for } r = 1, 2, \ldots, \frac{(k-1)M-20}{12},$$

$$\left(0_1, \left(\frac{5(k-1)M+8}{24}\right)_1, \left(\frac{5(k-1)M+32}{24}\right)_1\right), \left(0_1, \left(\frac{6(k-1)M+24}{24}\right)_1, \left(\frac{8(k-1)M+32}{24}\right)_1\right),$$

$$\left(0_1, \left(\frac{6(k-1)M+48}{24}\right)_1, \left(\frac{10(k-1)M+16}{24}\right)_1\right), \left(0_1, \left(\frac{8(k-1)M+8}{24}\right)_1, \left(\frac{(k-1)M}{2}\right)_1\right),$$

$$\left(0_0, \left(\frac{M}{2}+r\right)_1, \left(\frac{(k+1)M}{2}-r\right)_1\right) \text{ for } r = 1, 2, \ldots, \frac{M-2}{6},$$

$$\left(0_0, \left(\frac{2M-4}{6}+r\right)_1, \left(\frac{(3k+2)M+2}{6}-r\right)_1\right) \text{ for } r = 1, 2, \ldots, \frac{M-2}{6}-1,$$

$$\left(0_0, 0_1, \left(\frac{(k-1)M+4}{6}\right)_1\right), \left(0_0, \left(\frac{2M-4}{6}\right)_1, \left(\frac{3kM-6}{6}\right)_1\right), \left(0_0, \left(\frac{2M-10}{6}\right)_1, \left(\frac{3kM}{6}\right)_1\right),$$

$$\left(0_0, \left(\frac{M+4}{12}\right)_1, \left(\frac{(6k-3)M-24}{12}\right)_1\right), \left(0_0, \left(\frac{(3k-2)M+4}{6}\right)_1, \left(\frac{(12k-9)M-12}{12}\right)_1\right),$$

$$\left(0_0, \left(\frac{(24k-17)M-56}{24}\right)_1, \left(\frac{(36k-29)M-32}{24}\right)_1\right),$$

$$\left(0_0, \left(\frac{(3k-2)M+4}{6}+r\right)_1, \left(\frac{(6k-4)M-10}{6}-r\right)_1\right) \text{ for } r = 0, 1, 2, \ldots, \frac{M-8}{24}-1,$$

$$\left(0_0, \left(\frac{(6k-3)M-24}{12}-r\right)_1, \left(\frac{(12k-9)M-12}{12}+r\right)_1\right) \text{ for } r = 1, 2, \ldots, \frac{M-8}{24}-2$$

$$\left(0_0, (r)_1, \left(\frac{(3k-2)M+4}{6}-r\right)_1\right) \text{ for } r = 1, 2, \ldots, \frac{M-8}{12}, \text{ and } \left(\infty, 0_1, \left(\frac{kM}{2}\right)_1\right).$$

In each case, the given collection of blocks, along with a collection of base blocks for a 1-rotational $STS(M+1)$ on the point set $\{\infty\} \bigcup \mathbb{Z}_M \times \{0\}$ under the automorphism $(\infty)(0_0, 1_0, \ldots, (M-1)_0)$, forms a collection of base blocks for a STS of the desired type. ∎

Lemmas 3.1–3.5 combine to give us necessary and sufficient conditions for the existence of the desired type of STS.

Theorem 3.1 *A $STS(v)$ admitting an automorphism of type $[\pi] = [1, 0, \ldots, 0, 1, 0, \ldots, 0, 1, 0, \ldots, 0]$ where $\pi_1 = \pi_M = \pi_N = 1$, $M < N$, exists if and only if $M \equiv 2$ or 8 $(mod\ 24)$ and $N = kM$ where*

1. *if $M \equiv 2$ $(mod\ 24)$ then $k \equiv 2, 3, 6$ or 11 $(mod\ 12)$,*
2. *if $M \equiv 8$ $(mod\ 24)$ then $k \equiv 0$ or 2 $(mod\ 3)$.*

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