Ch. 9
Center of Mass
Momentum

Question 6
Problems: 3, 19, 21, 27, 31, 35, 39, 49, 51, 55, 63, 69, 71, 77
Center of Mass

Use center of mass when no longer dealing with a point particle.

“The center of mass of a system of particles is the point that moves as though:
1) all of the system’s mass were concentrated there and
2) all the external forces were applied there.”

Basically, we treat a complex object as a point particle that is located at the objects center of mass.
examples:

Throw a ball at an angle, it follows a parabolic trajectory.

Flip an object such as a hammer, baseball bat, or tennis racket. Different points on the object will follow different paths. The center of mass of the object will follow a parabolic path.

Simple way to find center of mass. Balance an object on a sharp edge. The center of mass is above the balance point.
Another way to find the center of mass.

Let the object hang freely. Draw a plumb line and trace it on the object.

Hang the object freely from another point and repeat. The point where the plumb lines intersect is the center of mass.

http://en.wikipedia.org/wiki/Center_of_mass

Note: the center of mass does not have to be where there is any mass.

Examples: doughnut, horseshoe, an empty water bottle
How to calculate the center of mass

The center of mass is the sum of the products of individual segments of mass and their respective locations, divided by the total mass.

Assume we have a system of two discrete particles in 1-D.

$m_1$ is at $x_1$

$m_2$ is at $x_2$

$$x_{\text{com}} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} = \frac{m_1 x_1 + m_2 x_2}{M}$$

You can see that if the two particles have the same mass, the CoM is halfway in between the particles.
Quite often there are more than 2 particles.

\[ x_{\text{com}} = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3 + \ldots}{m_1 + m_2 + m_3 + \ldots} = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i} = \frac{\sum_{i=1}^{n} m_i x_i}{M} \]

If particles in the system are in 3-D.

\[ x_{\text{com}} = \frac{1}{M} \sum_{i=1}^{n} m_i x_i \quad y_{\text{com}} = \frac{1}{M} \sum_{i=1}^{n} m_i y_i \quad z_{\text{com}} = \frac{1}{M} \sum_{i=1}^{n} m_i z_i \]

(These are scalar equations.)
Using position vector:

\[ \vec{r}_i = x_i \hat{i} + y_i \hat{j} + z_i \hat{k} \]

\[ \vec{r}_{com} = x_{com} \hat{i} + y_{com} \hat{j} + z_{com} \hat{k} \]

We can replace the scalar equation with a vector equation.

\[ \vec{r}_{com} = \frac{1}{M} \sum_{i=1}^{n} m_i \vec{r}_i \]
Solid bodies
(continuous mass distributions)

When the mass is distributed continuously, we replace the summations with integrals.

Break up object into different mass elements each with mass $dm$.

$$x_{com} = \frac{1}{M} \int x dm$$  $$y_{com} = \frac{1}{M} \int y dm$$  $$z_{com} = \frac{1}{M} \int z dm$$
If the density is not constant we can still find the center of mass.

If we know how it is distributed, \( m(x) \) we can use the integration method to find the CoM.

\[
\text{dm} = \rho \, dV \quad \bar{r}_{\text{com}} = \frac{1}{M} \int r \, dm = \frac{1}{M} \int \rho(r) r \, dV
\]
For the sake of simplicity we will only work with objects of uniform density.

\[ \rho = \frac{dm}{dV} = \frac{M}{V} \]

Substituting \( dm = (M/V)dv \), we get:

\[ x_{com} = \frac{1}{V} \int x dV \quad y_{com} = \frac{1}{V} \int y dV \quad z_{com} = \frac{1}{V} \int z dV \]
Newton’s 2\textsuperscript{nd} Law for a system of particles.

If we have multiple interacting particles, we want to study the center of mass.

Slide one hockey puck on a frictionless surface into an identical puck. Before they collide the CoM is moving with an initial velocity.

We will see that after they collide the CoM will maintain its velocity.
\[ \vec{F}_{net} = M \vec{a}_{com} \]

1) \( F_{net} \) is the net external force acting on the system. In the hockey puck example, since the pucks were the system, the forces they exert on each other are internal.

2) \( M \) is the total mass

3) \( a_{com} \) is the acceleration of the center of mass.

If we want we can break this equation into its components.

\[
F_{net,x} = M \ a_{com,x} \quad F_{net,y} = M \ a_{com,y} \quad F_{net,z} = M \ a_{com,z}
\]
Applications of studying CoM

1) Ballet dancer – as dancer jumps, raises arms and legs to raise the center of mass. The center of mass follows the parabolic trajectory. The movement of the CoM decreases the height that is attained by the head and torso, producing the illusion that the dancer floats. See pictures on page 207.

2) High jumper – As the high jumper goes over the bar the body is arched. The center of mass doesn’t have to go over the bar.
Linear Momentum

Linear momentum is different than angular momentum which is in a later chapter. In this chapter, whenever momentum is mentioned, it is linear.

Momentum is the product of the mass and velocity

\[ p = m \, v \]

Momentum is a vector quantity (direction is important).

Momentum is in the same direction as the object’s velocity.

The units of momentum are: \( \text{kg m/s} \)
Examples

A 1000 kg truck moving with velocity 30 m/s
\[ p = (1000 \text{ kg})(30 \text{ m/s}) = 3 \times 10^4 \text{ kg m/s} \]

A 3000 kg truck moving with velocity 30 m/s
\[ p = (3000 \text{ kg})(30 \text{ m/s}) = 9 \times 10^4 \text{ kg m/s} \]

A 0.14 kg baseball with velocity 40 m/s
\[ p = (0.14 \text{ kg})(40 \text{ m/s}) = 5.6 \text{ kg m/s} \]

A 6 kg bowling ball with velocity 5 m/s
\[ p = (6 \text{ kg})(4 \text{ m/s}) = 24 \text{ kg m/s} \]
Momentum

\[ \textbf{F}_{\text{net}} = m\textbf{a} \quad \text{and:} \quad \textbf{a} = \frac{\Delta \textbf{v}}{\Delta t} \]

\[ \textbf{F}_{\text{net}} = m \frac{\Delta \textbf{v}}{\Delta t} \]

Use momentum \( \textbf{p} = m\textbf{v} \)

If the mass is constant:

\[ \frac{d\textbf{p}}{dt} = m \frac{d\textbf{v}}{dt} \]

So:

\[ \textbf{F}_{\text{net}} = \frac{\Delta \textbf{p}}{dt} = \frac{d\textbf{p}}{dt} \]

The linear momentum of a system of particles can only change if there is a net external force.
Collision and impulse

\[ \vec{F}_{\text{net}} = \frac{\Delta \vec{p}}{dt} = \frac{d\vec{p}}{dt} \]

If force is time dependant:

\[ d\vec{p} = \vec{F}(t)dt \]

\[ \int_{t_i}^{t_f} d\vec{p} = \int_{t_i}^{t_f} \vec{F}(t)dt \]

\[ \vec{p}_f - \vec{p}_i = \Delta \vec{p} = \vec{J} = \int_{t_i}^{t_f} \vec{F}(t)dt \]

\( \vec{J} \) is referred to as the impulse. (change in momentum)
Force vs. time graph

The area under a Force vs. time graph tells us the impulse.

If we don’t know how the force varies but we know the average force we can find the impulse.

\[ J = F_{\text{ave}} \Delta t \]
Newton’s 3\textsuperscript{rd} Law

Since forces occur in equal and opposite pairs so do impulses.

When a bat hits a ball. The bat exerts an impulse on the ball. The ball exerts an impulse with the same magnitude but opposite direction on the bat.
Momentum is a vector and can be split into components.

For an isolated system, if a component of the momentum is zero, then the component of $\Delta p$ in that direction will also be zero.

If a system is not isolated, $\Delta p$ would be nonzero.

The key is that determining what makes up the system is important.
Conservation of Momentum

When there is no net external force acting on a system, the total momentum of the system remains constant with time.

When a collision occurs in an isolated system, the individual momentums of objects may change, but the total momentum vector of the whole system is a constant.

For a system with two particles:

\[ m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f} \]
An archer stands on frictionless ice and shoots a 0.5kg arrow horizontally at 50 m/s. The combined mass of the archer and the bow is 60 kg. With what velocity will the archer move after shooting the arrow?

Since the system is the archer and the arrows, it is isolated. The forces that the bow and arrow exert on each other are internal forces.

Use conservation of momentum

\[ m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f} \]
Archer
Initially the total momentum is zero. The initial velocities of the archer and the arrow are both zero.

So after the arrow is shot, the total momentum must still be zero.
Archer
(labeled the archer as 1 and the arrow as 2)

\[ m_1v_{1i} + m_2v_{2i} = m_1v_{1f} + m_2v_{2f} \]

\[
(60\text{kg})0\text{m/s} + (0.5\text{kg})0\text{m/s} = (60\text{kg})v_{1f} + (0.5\text{kg})(50\text{m/s})
\]

\[
0 = (60\text{kg})v_{1f} + (0.5\text{kg})(50\text{m/s})
\]

\[
(60\text{kg})v_{1f} = - (0.5\text{kg})(50\text{m/s})
\]

\[
v_{1f} = -0.417 \text{ m/s}
\]

If the arrow is shot to the right, the archer moves to the left.
Collisions

• When objects collide, momentum is always conserved.

• Kinetic energy is not always conserved. Some of the kinetic energy is turned into sound, work needed to deform the object, heat, internal energy.
Collisions

• We will be looking at two types of collisions.

• Perfectly elastic collisions
  – momentum and kinetic energy is conserved
  – the objects leave each other with the same speed as they hit each other
  – there is no deformation of the objects during the collision
  – examples:
    • Pool balls (approximately)
    • Proton-proton collision
Collisions

• Perfectly Inelastic collisions
  – momentum is conserved, but kinetic energy is lost
  – the objects stick together after the collision. (they move with the same velocity)
  – examples:
    • Two cars hitting each other and the bumpers lock up
    • Football player being tackled by another player
    • Person catching a ball
    • Shooting a ball into a ballistic pendulum
Collisions

Nearly all collisions in the real world fall somewhere in between being perfectly elastic and perfectly inelastic.

When you hit a baseball the ball doesn’t stick to the bat, so the collision isn’t perfectly inelastic.

Some of the kinetic energy is converted to sound and is used to momentarily deform the ball and bat, so the collision isn’t perfectly elastic either.
Perfectly Inelastic Collisions

Since the two objects have the same velocity after the collision, \( v_{1f} = v_{2f} = v_f \)

we can rewrite the conservation of energy as:

\[
m_1 v_{1i} + m_2 v_{2i} = (m_1 + m_2)v_f
\]

Solving for the final velocity gives us:

\[
v_f = \frac{m_1 v_{1i} + m_2 v_{2i}}{m_1 + m_2}
\]

Remember to treat \( v_{1i} \) and \( v_{2i} \) as vectors. Their direction matters.
Inelastic Collision

 Shoot a 0.05 kg bullet at 300 m/s into a 5kg wooden block. The bullet gets stuck in the block, so they move together afterwards. Find the final velocity of the bullet/block.

Cons. of Mom. says:  

\[ m_1 v_{1i} + m_2 v_{2i} = (m_1 + m_2) v_f \]

Let the bullet be mass 1 and the block be mass 2.

\[ v_f = \frac{(0.05 \text{ kg})(300 \text{ m/s}) + (5 \text{ kg})(0 \text{ m/s})}{0.05 \text{ kg} + 5 \text{ kg}} = 2.97 \text{ m/s} \]
Perfectly Elastic Collision

Momentum is conserved
\[ m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f} \]

Kinetic Energy is conserved
\[ \frac{1}{2} m_1 v_{1i}^2 + \frac{1}{2} m_2 v_{2i}^2 = \frac{1}{2} m_1 v_{1f}^2 + \frac{1}{2} m_2 v_{2f}^2 \]

Combining these two equations, if I gave you the initial velocities of the objects you could solve for the two final velocities simultaneously.
Elastic Collisions

Do a bunch of algebra and you will get:

\[ v_{1f} = \frac{(m_1 - m_2)v_{1i} + 2m_2v_{2i}}{m_1 + m_2} \]

and

\[ v_{2f} = \frac{2m_1v_{1i} + (m_2 - m_1)v_{2i}}{m_1 + m_2} \]
Special cases for elastic collisions

\[ v_{1f} = \frac{(m_1 - m_2)v_{1i} + 2m_2v_{2i}}{m_1 + m_2} \quad v_{2f} = \frac{2m_1v_{1i} + (m_2 - m_1)v_{2i}}{m_1 + m_2} \]

If the target \((m_2)\) is stationary, \(v_{2i} = 0\)

\[ v_{1f} = \frac{(m_1 - m_2)v_{1i}}{m_1 + m_2} \quad v_{2f} = \frac{2m_1v_{1i}}{m_1 + m_2} \]
\[ v_{1f} = \frac{(m_1 - m_2)v_{1i} + 2m_2v_{2i}}{m_1 + m_2} \quad \text{and} \quad v_{2f} = \frac{2m_1v_{1i} + (m_2 - m_1)v_{2i}}{m_1 + m_2} \]

If the two masses are equal we get:

\[ V_{1f} = v_{2i} \quad \text{and} \quad v_{2f} = v_{1i} \]

If the target is stationary and the two masses are equal we get:

\[ V_{1f} = 0 \quad \text{and} \quad v_{2f} = v_{1i} \]

If the target is stationary and the target is massive \((m_2 >> m_1)\) we get:

\[ v_{1f} \approx -v_{1i} \quad \text{and} \quad v_{2f} \approx \frac{2m_1}{m_2}v_{1i} \]

If the target is stationary and the projectile is massive \((m_1 >> m_2)\) we get:

\[ v_{1f} \approx v_{1i} \quad \text{and} \quad v_{2f} \approx 2v_{1i} \]
Some visual examples of elastic collisions


Do sample problem 9-11
Collisions in more than 1-Dimension

Remember that momentum is a vector. We can expand our work into 2 dimensions.

All you need to realize is that the momentum in the x-direction is conserved.
In addition, the momentum in the y-direction is conserved.

The next example is a projectile hitting a target with an elastic, glancing collision (not head on) with a target.
X-direction:  $m_1 v_{1i} = m_1 v_{1f} \cos \theta_1 + m_2 v_{2f} \cos \theta_2$

Y-direction:  $0 = -m_1 v_{1f} \sin \theta_1 + m_2 v_{2f} \sin \theta_2$

In addition we write from conservation of energy:

$$\frac{1}{2} m_1 v_{1i}^2 = \frac{1}{2} m_1 v_{1f}^2 + \frac{1}{2} m_2 v_{2f}^2$$

There are 7 variables:  $m_1, m_2, v_{1i}, v_{1f}, v_{2f}, \theta_1, \theta_2$.

If we know 4 of them we can find the other 3.

http://www.youtube.com/watch?v=KteP6k1wg94
Ballistic Pendulum

A 0.05 bullet with velocity 150 m/s is shot into a 3 kg ballistic pendulum. Find how high the pendulum rises after the bullet gets stuck inside.
Ballistic Pendulum

First use conservation of momentum to find the final velocity of the bullet/pendulum.

This is an inelastic collision.

\[ m_1 v_{1i} + m_2 v_{2i} = (m_1 + m_2) v_f \]

\[ (0.05\text{kg})(150\text{m/s}) + (3\text{kg})(0\text{m/s}) = (3.05\text{kg})v_f \]

\[ v_f = 2.5 \text{ m/s} \]
Ballistic Pendulum

Now that we know the ballistic pendulum with the bullet in it begins to swing with a speed of 2.5 m/s, we use conservation of energy to find how high it swings.

\[
\frac{1}{2} mv^2 = mg\Delta y
\]

\[
\frac{1}{2} m (2.5\text{m/s})^2 = mg\Delta y
\]

\[
\Delta y = 0.32 \text{ m}
\]
Systems with varying mass

Example where the mass varies is rocket motion. As the rocket burns fuel, its mass is decreased.

Gases are expelled at a high velocity out the bottom of the rocket. Since the rocket and the expelled gas exert equal and opposite forces on each other (internal forces), the total momentum is conserved.

To conserve the momentum, the rocket goes in the direction opposite to the relative velocity of the gas.
1) First rocket equation:

$$R v_{rel} = M a$$

- $R$ – instantaneous rate of fuel consumption
- $V_{rel}$ – relative speed of gas product to rocket
- $M$ – instantaneous mass
- $a$ – instantaneous acceleration

Left hand side of equation has units of force

$R v_{rel}$ is the thrust (T) of the engine.
The rocket accelerating so its velocity is non-constant.
By equating the momentum carried away by the gas to the momentum of the rocket we get:

\[- \frac{dM}{dt} v_{rel} = M \frac{dv}{dt}\]

\[dv = -v_{rel} \frac{dM}{M}\]

\[\int_{v_i}^{v_f} dv = -v_{rel} \int_{M_i}^{M_f} \frac{dM}{M}\]

\[v_f - v_i = v_{rel} \ln \frac{M_i}{M_f}\]

2\textsuperscript{nd} Rocket equation

An ideal rocket would reach the destination, with only the payload and this would be the most efficient case. (Maximize $M_i/M_f$)
Problems: 4, 10, 36, 41, 54, 78