

PHYS-4007/5007: Computational Physics
Course Lecture Notes
Section X

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Version 5.0

Abstract

These class notes are designed for use of the instructor and students of the course **PHYS-4007/5007: Computational Physics** taught by Dr. Donald Luttermoser at East Tennessee State University.

X. Trajectories and Orbits

A. Introduction.

1. This section concerns itself with the numerical solution to Newton's Second Law of Motion in a gravitational potential field.
2. As we saw with Eq. (IX-1), Newton's Second Law of Motion can be written as:

$$F_{\text{tot}} = \sum_{i=1}^N F [t, r(t), v(t)] = ma = m \frac{dv(t)}{dt} = m \frac{d^2 r(t)}{dt^2}, \quad (\text{X-1})$$

where here we are using the standard notation of $r \equiv$ displacement, $v \equiv$ linear velocity, $a \equiv$ linear acceleration, $t \equiv$ time, $m \equiv$ mass of the object, and N is the total number of independent forces that are acting on the body.

- a) The position $r(t)$ of a particle of mass m is acted upon by the net force F_{tot} , which may be a function of time t , position $r(t)$, and the velocity $v(t) = dr(t)/dt$.
- b) The motion of the object can then be described completely by specifying F_{tot} and setting initial conditions at t_0 to r and v .
- c) Note that derivatives with respect to time are often written with the 'dot' notation:

$$v = \frac{dr(t)}{dt} = \dot{r} \quad (\text{for velocity})$$
$$a = \frac{dv(t)}{dt} = \frac{d^2 r(t)}{dt^2} = \ddot{r} \quad (\text{for acceleration})$$

so that Newton's 2nd law can be written as $F = m\ddot{r}$.

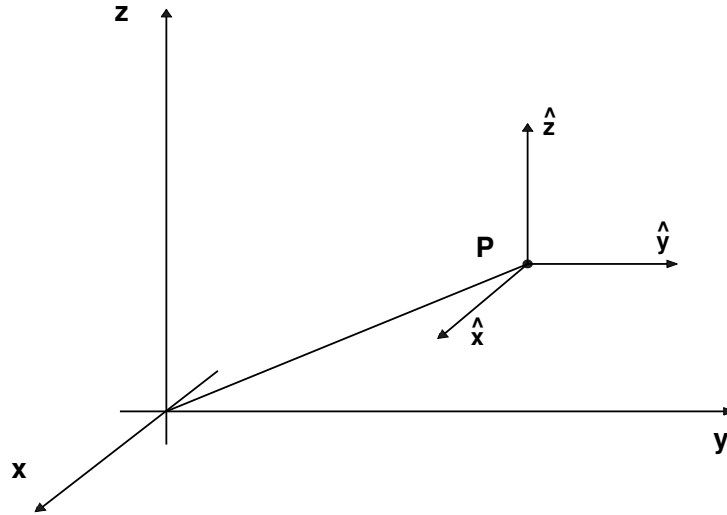


Figure X-1: The orthogonal (Cartesian) coordinate system.

3. In classical mechanics, the *static* potential field is related to a **conservative** force by the equation

$$\vec{F} = -\vec{\nabla}U , \quad (\text{X-2})$$

where U is the potential energy of the field (a scalar), and the “del” operator acting on a scalar is referred to as taking the **gradient** of the scalar \implies it converts a scalar to a vector. The “del” operator takes first derivatives on each coordinate of the vector space. As such, it has a different form depending on the coordinate system:

- a) Orthogonal (Cartesian) Coordinates (x, y, z) :

$$\vec{\nabla} = \hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} \frac{\partial}{\partial z} , \quad (\text{X-3})$$

such that

$$\vec{\nabla}f(x, y, z) = \hat{x} \frac{\partial f}{\partial x} + \hat{y} \frac{\partial f}{\partial y} + \hat{z} \frac{\partial f}{\partial z} .$$

- b) Spherical-Polar (Spherical) Coordinates (r, θ, ϕ) :

$$\vec{\nabla} = \hat{r} \frac{\partial}{\partial r} + \hat{\theta} \frac{1}{r} \frac{\partial}{\partial \theta} + \hat{\phi} \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} . \quad (\text{X-4})$$

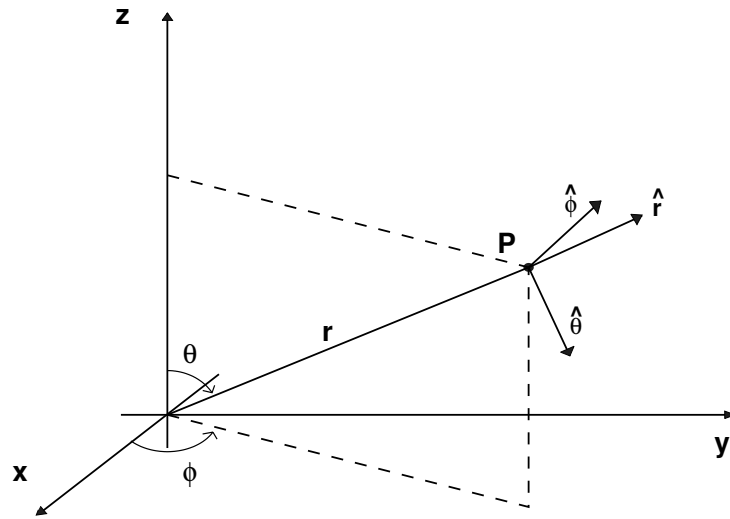


Figure X-2: The spherical-polar coordinate system.

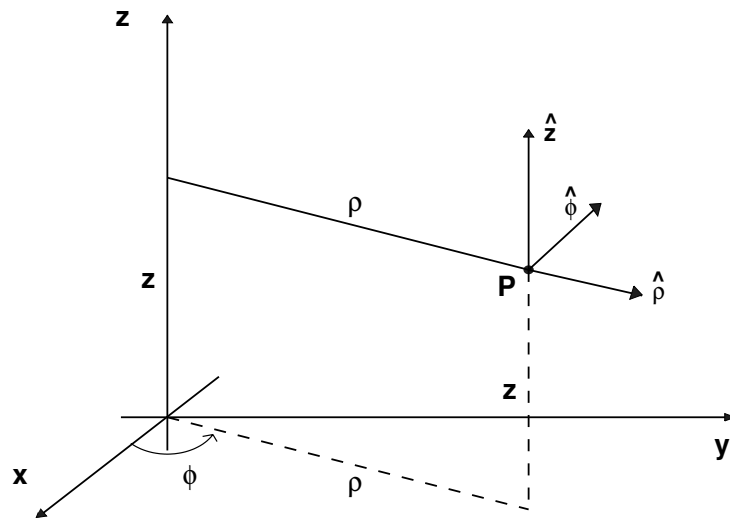


Figure X-3: The circular-cylindrical coordinate system.

c) Circular-Cylindrical (Cylindrical) Coordinates (ρ, ϕ, z) :

$$\vec{\nabla} = \hat{\rho} \frac{\partial}{\partial \rho} + \hat{\phi} \frac{1}{\rho} \frac{\partial}{\partial \phi} + \hat{z} \frac{\partial}{\partial z} . \quad (\text{X-5})$$

4. In this section, we will be using the gravitational potential field to describe the potential energy, then from Newton's Universal Law of Gravity,

$$\vec{F}_g(r) = -\frac{G m_1 m_2}{r^2} \hat{r} , \quad (\text{X-6})$$

we see that this force only depends upon r and is negative since it points in the opposite direction of \hat{r} due to its attractive nature. In this equation the m 's represent the masses of bodies 1 and 2 which are separated by a distance r . $G = 6.673 \times 10^{-11} \text{ N m}^2/\text{kg}^2 = 6.673 \times 10^{-8} \text{ dyne cm}^2/\text{g}^2$ is the Universal Gravitation Constant.

a) Using Eq. (X-6) in conjunction with Eq. (X-4), we immediately see that $U = U(r)$ since $F = F(r)$.

i) To solve the gradient equation for the potential energy (Eq. X-2), we can take the dot product (*i.e.*, inner product) of both sides of the equation of Eq. (X-2) with the differential displacement $d\vec{r}$ and integrate from the initial (1) to the final position (2).

ii) Since the force here is conservative, the work done by this force is independent of the path or trajectory taken. Hence, we can use the standard definite integral instead of a path integral to solve for U :

$$\int_1^2 \vec{F} \cdot d\vec{r} = - \int_1^2 \vec{\nabla} U \cdot d\vec{r} \quad (\text{X-7})$$

$$- \int_1^2 \left(\frac{G m_1 m_2}{r^2} \hat{r} \right) \cdot (dr \hat{r}) = - \int_1^2 \left(\frac{\partial U}{\partial r} \hat{r} \right) \cdot (dr \hat{r})$$

$$\int_1^2 \frac{G m_1 m_2}{r^2} dr = \int_1^2 \frac{dU}{dr} dr \quad (\text{X-8})$$

where the partials become full derivatives since $U = U(r)$.

iii) Integrating Eq. (X-8), we get

$$-\frac{G m_1 m_2}{r} \Big|_1^2 = \int_1^2 dU = U_2 - U_1 ,$$

or

$$U_1 - U_2 = G m_1 m_2 \left(\frac{1}{r_2} - \frac{1}{r_1} \right) . \quad (\text{X-9})$$

b) From Eq. (X-9), we can immediately see that the potential energy for a gravitational field takes on the form

$$U = -\frac{G m_1 m_2}{r} . \quad (\text{X-10})$$

i) From this equation we see that $U \rightarrow 0$ as $r \rightarrow \infty$.

ii) Also we can see that gravitational potential energy is a negative energy.

5. Besides potential energy, any body in motion has a *kinetic* energy associated with it.

a) The left-hand side of Eq. (X-7) is the definition of the work done on a particle by a force field:

$$W \equiv \int_1^2 \vec{F} \cdot d\vec{r} = U_1 - U_2 . \quad (\text{X-11})$$

b) Using one of the forms of Newton's Second Law of Motion in Eq. (X-1) and rewriting $d\vec{r}$ as $(d\vec{r}/dt)dt$, we can write

$$\vec{F} \cdot d\vec{r} = \left(m \frac{d\vec{v}}{dt} \right) \cdot \left(\frac{d\vec{r}}{dt} dt \right) = m \frac{d\vec{v}}{dt} \cdot \vec{v} dt .$$

Note that

$$\frac{d}{dt}(\vec{v} \cdot \vec{v}) = \vec{v} \cdot \frac{d\vec{v}}{dt} + \frac{d\vec{v}}{dt} \cdot \vec{v} = 2 \frac{d\vec{v}}{dt} \cdot \vec{v} ,$$

so

$$\frac{d\vec{v}}{dt} \cdot \vec{v} = \frac{1}{2} \frac{d}{dt} (\vec{v} \cdot \vec{v})$$

and hence,

$$\begin{aligned} \vec{F} \cdot d\vec{r} &= \frac{1}{2} m \frac{d}{dt} (\vec{v} \cdot \vec{v}) dt = \frac{1}{2} m \frac{d}{dt} (v^2) dt \\ &= d\left(\frac{1}{2}mv^2\right) . \end{aligned} \quad (\text{X-12})$$

- c) Now using Eq. (X-12) in the first part of Eq. (X-11), we get

$$\begin{aligned} W \equiv \int_1^2 \vec{F} \cdot d\vec{r} &= \int_1^2 d\left(\frac{1}{2}mv^2\right) = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2 \\ &= T_2 - T_1 , \end{aligned} \quad (\text{X-13})$$

where ‘ T ’ represents kinetic energy.

- d) Inserting this value for the work in Eq. (X-11), we see

$$W = T_2 - T_1 = U_1 - U_2$$

or

$$\begin{aligned} T_1 + U_1 &= T_2 + U_2 \\ E_1 &= E_2 , \end{aligned} \quad (\text{X-14})$$

where E represents the total mechanical energy of the system.

- e) Eq. (X-14) is called the **conservation of mechanical energy**, which is valid only for a conservative force \rightarrow forces that do not depend on time nor does the work done by such a force depend upon the path taken in a trajectory.

6. Whereas the force is equal to the negative gradient of the potential energy (see Eq. X-2), the acceleration due to a conservative

force can be determined from the **potential** Φ of the force ('potential' is the potential energy per unit mass). For gravity

$$\vec{g} \equiv -\vec{\nabla}\Phi , \quad (\text{X-15})$$

thus

$$\Phi = \frac{U}{m} = -\frac{GM}{r} . \quad (\text{X-16})$$

Note this could also be proven by following the technique shown in Eqs. (X-7) through (X-10). To determine the gravitational potential, one must know the manner in which mass is distributed throughout a body.

- a) The potential due to a continuous distribution of matter is

$$\Phi = -G \int_V \frac{\rho(\vec{r}')}{r'} dv' , \quad (\text{X-17})$$

(*i.e.*, a volume integral) where $\rho(\vec{r}')$ is the mass density over volume element dv' at distance r' from the origin integrated over the entire volume V .

- b) If mass is distributed over a thin shell (*i.e.*, a surface distribution), then

$$\Phi = -G \int_S \frac{\rho_s}{r'} da' , \quad (\text{X-18})$$

(*i.e.*, a surface integral) where ρ_s is the surface density of mass (areal mass density), da' is an area differential, and the integral is taken over the surface S .

- c) Finally, if there is a line source with linear mass density ρ_ℓ of length L ,

$$\Phi = -G \int_L \frac{\rho_\ell}{r'} ds' , \quad (\text{X-19})$$

(*i.e.*, a line integral) where ds' is a length differential on the line source of gravity.

7. The work done by a potential field is

$$\begin{aligned} dW &= -\vec{g} \cdot d\vec{r} = \vec{\nabla}\Phi \cdot d\vec{r} \\ &= \sum_{i=1}^N \frac{\partial\Phi}{\partial x_i} dx_i = d\Phi . \end{aligned} \quad (\text{X-20})$$

8. When dealing with motion in gravitational fields, there are two regimes that are typically encountered: (1) *trajectories* (near a surface of a large mass, *e.g.*, Earth) and (2) *orbits* (where 2 masses can be considered as point-like).

a) For orbits, we use the general form of the gravitational potential as described in Eqs. (X-10,16):

$$U_g = -\frac{GMm}{r} , \quad (\text{X-21})$$

where we are now using M (the larger mass) for m_1 and m (the smaller mass) for m_2 and

$$\Phi = -\frac{GM}{r} . \quad (\text{X-22})$$

b) For trajectories, typically the maximum height ($y_{\max} = h$) reached is small with respect to R_{\oplus} and hence $g \approx$ constant. As such, we can write Eqs. (X-9 and X-11) as

$$W = U_1 - U_2 = GM_{\oplus}m \left(\frac{1}{r_2} - \frac{1}{r_1} \right) .$$

i) If we take point ‘2’ to be the Earth’s surface and ‘1’ to be the position of the projectile, then

$$\begin{aligned} \Delta U &= G M_{\oplus} m \left(\frac{1}{R_{\oplus}} - \frac{1}{R_{\oplus} + h} \right) \\ &= G M_{\oplus} m \left(\frac{R_{\oplus} + h}{R_{\oplus} (R_{\oplus} + h)} - \frac{R_{\oplus}}{R_{\oplus} (R_{\oplus} + h)} \right) \\ &= G M_{\oplus} m \left(\frac{R_{\oplus} + h - R_{\oplus}}{R_{\oplus} (R_{\oplus} + h)} \right) \\ &= G M_{\oplus} m \left(\frac{h}{R_{\oplus} (R_{\oplus} + h)} \right) . \end{aligned}$$

- ii) If R_{\oplus} is much greater than h (which it will be for experiments near the Earth's surface), $h \ll R_{\oplus}$. As such, $R_{\oplus} + h \approx R_{\oplus}$ and the equation above becomes

$$\begin{aligned} \Delta U &= G M_{\oplus} m \left(\frac{h}{R_{\oplus} (R_{\oplus})} \right) = G M_{\oplus} m \left(\frac{h}{R_{\oplus}^2} \right) \\ &= \frac{G M_{\oplus} m h}{R_{\oplus}^2} = m \frac{G M_{\oplus}}{R_{\oplus}^2} h . \end{aligned} \quad (\text{X-23})$$

- iii) Using Eqs. (X-15 and X-16), we see that

$$\begin{aligned} \vec{g} &= -\vec{\nabla} \Phi = -\vec{\nabla} \left(\frac{G M_{\oplus}}{r} \right) \\ &= -\frac{d}{dr} \left(\frac{G M_{\oplus}}{r} \right) \hat{r} = \frac{G M_{\oplus}}{r^2} \hat{r} \end{aligned} \quad (\text{X-24})$$

and at the Earth's surface,

$$g = \frac{G M_{\oplus}}{R_{\oplus}^2} , \quad (\text{X-25})$$

where 'g' is referred to as the Earth's **surface gravity**.

- iv) Using Eq. (X-25) in Eq. (X-23), we finally get

$$\Delta U = mgh = mg\Delta y ,$$

where $\Delta y = y - y_o = h$ is just the change in height from our initial position y_o (typically the ground) and y is an arbitrary position in the trajectory above y_o .

- v) If we arbitrarily set $y_o = 0$, then the potential at that position is zero, and y represents the position above the ground (y_o). As such, the potential energy becomes

$$U = mgy . \quad (\text{X-26})$$

9. Trajectory calculations can be difficult due to non-gravitational forces that enter the calculations.

a) A **drag force** due to air friction which usually takes the form

$$\vec{F}_r = -mkv^n \frac{\vec{v}}{v}, \quad (\text{X-27})$$

where \vec{F}_r represents the retarding (*i.e.*, drag) force, v is the magnitude of the velocity, \vec{v} is the velocity vector (hence \vec{F}_r is in the opposite direction of \vec{v} due to the negative sign, note that the ratio \vec{v}/v is essentially just a unit vector in the direction of \vec{v}), and k is the drag coefficient. As such, the total force on the object now becomes

$$\vec{F} = \vec{F}_g + \vec{F}_r. \quad (\text{X-28})$$

b) If the downrange distance of the projectile is large enough such that the Earth's surface can no longer be represented as a flat plane, the Earth's rotation has to be taken into account. To an observer in the rotating coordinate system, the effective force (ignoring air friction) is

$$\vec{F}_{\text{eff}} = m\vec{a}_f - m\vec{\omega} \times (\vec{\omega} \times \vec{r}) - 2m\vec{\omega} \times \vec{v}_r. \quad (\text{X-29})$$

i) $\vec{F}_f = m\vec{a}_f$ is the force in the *fixed* coordinate system (which is just Newton's 2nd law). This force is said to be an *inertial* force since it applies only to a static coordinate system.

ii) $\vec{F}_{\text{cf}} = -m\vec{\omega} \times (\vec{\omega} \times \vec{r})$ is the **centrifugal force**, which results from trying to write an inertial force law for a noninertial (*i.e.*, *accelerating*) reference frame (note that $\vec{\omega}$ is called the angular velocity). The minus sign in this term implies that this *pseudo*-force (see below) is directed outwards from the cen-

ter of rotation (perpendicular to the axis of rotation).

iii) $\vec{F}_{\text{cor}} = -2m\vec{\omega} \times \vec{v}_r$ is the **Coriolis force**, which results from the Earth's rotation ($\vec{v}_r \equiv$ Earth's rotational velocity) \implies this “force” arises when an attempt is made to describe motion relative to a rotating body (*i.e.*, the ground moves out from under you when the projectile is in the air).

iv) Note that the centrifugal and Coriolis forces are not *forces* in the usual sense of the word. They are only introduced so that the *inertial (non-accelerating) frame* equation

$$\vec{F} = m\vec{a}_f$$

(*i.e.*, Newton's 2nd law) can have a *non-inertial (accelerating) frame* analogous equation:

$$\vec{F}_{\text{eff}} = m\vec{a}_r ,$$

so that

$$\vec{F}_{\text{eff}} = m\vec{a}_f + (\text{non-inertial terms}).$$

B. Numerical Solutions for Trajectories.

1. Trajectories with $h \ll R_{\oplus}$ and $x \ll R_{\oplus}$.

a) Combining Eqs. (X-1, 27, & 28), Newton's 2nd law becomes:

$$\sum \vec{F} = \vec{F}_g + \vec{F}_r = m \frac{d^2 \vec{r}}{dt^2}$$

or

$$m \frac{d^2 \vec{r}}{dt^2} = -mg \hat{y} - mkv^n \frac{\vec{v}}{v} , \quad (\text{X-30})$$

where we have defined $+\hat{y}$ in the upward direction.

- b) Since the atmospheric drag force is a function of \vec{v} , it is more convenient to write Eq. (X-30) as

$$\begin{aligned} m \frac{d\vec{v}}{dt} &= -mg \hat{y} - mkv^n \frac{\vec{v}}{v} \\ \frac{d\vec{v}}{dt} &= -g \hat{y} - kv^n \frac{\vec{v}}{v} \end{aligned} \quad (\text{X-31})$$

- i) At low velocities, $n \approx 1$ and the magnitude of the drag force follows

$$F_r \approx -B_1 v \quad (\text{X-32})$$

\implies this is known as **Stoke's law**.

- ii) As v increases, $n \rightarrow 2$ and the drag force follows

$$F_r \approx -B_2 v^2 . \quad (\text{X-33})$$

- iii) As such, we can write a general form of the drag force as

$$F_r \approx -B_1 v - B_2 v^2 \quad (\text{X-34})$$

or

$$F_r = - \sum_{i=1}^N B_i v^i \quad (\text{X-35})$$

\implies a simple power series in v (note that $B_i \rightarrow 0$ faster than $v^i \rightarrow \infty$ as $i \rightarrow \infty$).

- iv) The B_i 's are related to the drag coefficient k in Eqs. (X-30 & 31).

- c) Since we are dealing with projectiles here, the drag force will simply be described by Eq. (X-33). As such, we need to solve each component of Eq. (X-31) — hence, we need to break the drag force into its component forces:

$$F_r = -mkv^2 .$$

i) Since

$$v^2 = v_x^2 + v_y^2 ,$$

we can write

$$F_{r,x} = F_r \cos \theta = F_r \frac{v_x}{v} , \quad (\text{X-36})$$

where θ is the angle between \vec{v}_x and \vec{v} .

ii) Likewise

$$F_{r,y} = F_r \sin \theta = F_r \frac{v_y}{v} . \quad (\text{X-37})$$

iii) Hence, the drag force has components of

$$\begin{aligned} F_{r,x} &= -mkvv_x \\ F_{r,y} &= -mkvv_y . \end{aligned} \quad (\text{X-38})$$

d) The x -component of Newton's 2nd law gives

$$\frac{dv_x}{dt} = -kvv_x \quad (\text{X-39})$$

with

$$\frac{dx}{dt} = v_x . \quad (\text{X-40})$$

e) The y -component gives

$$\frac{dv_y}{dt} = -g - kvv_y \quad (\text{X-41})$$

with

$$\frac{dy}{dt} = v_y . \quad (\text{X-42})$$

f) The solution to these first-order DEs can be found with simple forward-difference equations:

$$\begin{aligned} x_{i+1} &= x_i + v_{x,i} \Delta t \\ v_{x,i+1} &= v_{x,i} - kvv_{x,i} \Delta t \\ y_{i+1} &= y_i + v_{y,i} \Delta t \\ v_{y,i+1} &= v_{y,i} - g \Delta t - kvv_{y,i} \Delta t . \end{aligned} \quad (\text{X-43})$$

- g) Eqs. (X-43) can be then solved as an initial value problem as described in §IX with x_o , y_o , $v_{x,o}$, and $v_{y,o}$ supplied by the user.
- i) The Δt steps are chosen to give the errors that follow Eq. (VII-15).
- ii) Calculations are carried out until a certain $\tau =$ *total time* is reached or some condition of x_i , y_i , $v_{x,i}$, or $v_{y,i}$ is satisfied.
- h) But what is the drag coefficient k ?
- i) Since the projectile is trying to push air of mass dm_{air} out of the way, where

$$dm_{\text{air}} \approx \rho A v dt , \quad (\text{X-44})$$

where ρ is the density of the air and A is the frontal area, we can guess that k is a function of ρ [and possibly t if the object is rotating \rightarrow then $A = A(t)$].

- ii) $k(y = 0) = k_o$ is usually given (based on air tunnel measurements), and the following expression is used for $k(y)$:

$$k(y) = \frac{\rho(y)}{\rho_o} k_o , \quad (\text{X-45})$$

where ρ_o is the density of air when the k_o measurement was made.

- i) Now we need a description of $\rho(y)$!
- i) One could supply a data table of ρ as a function of height (see Appendix B.1 in *Fundamentals of Atmospheric Modeling* by Mark Jacobson, 1999, Cambridge University Press).

- ii) One could solve the following set of differential equations to determine $\rho(y)$:

$$P = Nk_B T \quad (\text{ideal gas law}) \quad (\text{X-46})$$

$$\rho = \sum N_i m_i \quad (\text{mass conservation}) \quad (\text{X-47})$$

$$\frac{dP}{dy} = -\rho g \quad (\text{hydrostatic equilibrium}) \quad (\text{X-48})$$

$$\frac{dT}{T} = \left(\frac{R}{c_p} \right) \frac{dP}{P} \quad (\text{Poisson's equation}) \quad (\text{X-49})$$

$$dQ = c_p dT - \alpha dP \quad (\text{energy conserv.}) \quad (\text{X-50})$$

where P = pressure, T = temperature, N = total particle density, ρ total mass density, N_i = number density of species i , m_i = mass of species i , k_B = Boltzmann's constant, g = surface gravity, R = universal gas constant, c_p = specific heat of air at constant pressure, Q = total heat (determined from solar radiation incident on the Earth's atmosphere), and α = specific volume of air.

- iii) Note that the solution to these equation would still only be an approximation since we have left out condensation, evaporation, sublimation, chemical reactions, and wind from these equations.

2. Trajectories with $h \sim R_\oplus$ and $x \sim R_\oplus$.

a) Rotating Coordinate Systems.

- i) Let's consider 2 sets of coordinate axes:
 \implies one 'fixed' = **inertial frame** (the *primed* [t'] coordinates),
 \implies one 'rotating' with respect to the fixed system and possibly in linear motion with respect to the fixed frame = **noninertial frame**.

- ii) Let point ‘ P ’ be a point in space that can both be measured from either frames, then

$$\vec{r}' = \vec{R} + \vec{r}, \quad (\text{X-51})$$

where \vec{r}' is the radius vector of P in the fixed system, \vec{r} is the radius vector of P in the moving system, and \vec{R} locates the origin of the moving system with respect to the fixed system as shown in Figure X-4.

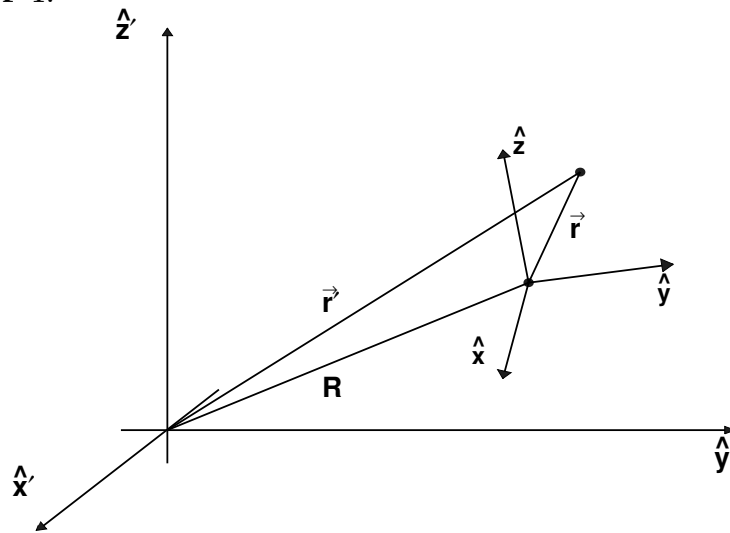


Figure X-4: Locating a point in an inertial frame and a noninertial frame.

- iii) The radius vector differential of the moving frame as measured in the fixed frame is related to the rotation angle differential of the noninertial frame:

$$(d\vec{r})_{\text{fixed}} = d\vec{\theta} \times \vec{r}, \quad (\text{X-52})$$

where here, the LHS of the equation is measured in the fixed frame and the RHS is measured in the rotating frame.

- iv) The time rate of change of \vec{r} as measured in the fixed frame is

$$\left(\frac{d\vec{r}}{dt}\right)_{\text{fixed}} = \frac{d\vec{\theta}}{dt} \times \vec{r} = \vec{\omega} \times \vec{r}, \quad (\text{X-53})$$

since the angular velocity $\vec{\omega}$ is defined by

$$\vec{\omega} \equiv \frac{d\vec{\theta}}{dt}. \quad (\text{X-54})$$

- v) If point P has a velocity $(d\vec{r}/dt)_{\text{rot}}$ with respect to the rotating system, this velocity must be added to $\vec{\omega} \times \vec{r}$ to obtain the time rate of change of \vec{r} in the fixed system:

$$\left(\frac{d\vec{r}}{dt}\right)_{\text{fixed}} = \left(\frac{d\vec{r}}{dt}\right)_{\text{rot}} + \vec{\omega} \times \vec{r}. \quad (\text{X-55})$$

- vi) This expression is not just limited to the displacement vector \vec{r} , in fact, for any arbitrary vector \vec{Q} , we have

$$\boxed{\left(\frac{d\vec{Q}}{dt}\right)_{\text{fixed}} = \left(\frac{d\vec{Q}}{dt}\right)_{\text{rot}} + \vec{\omega} \times \vec{Q}}. \quad (\text{X-56})$$

- vii) Note that the angular acceleration $\vec{\dot{\omega}}$ is the same in both the fixed and rotating systems:

$$\begin{aligned} \left(\frac{d\vec{\omega}}{dt}\right)_{\text{fixed}} &= \left(\frac{d\vec{\omega}}{dt}\right)_{\text{rot}} + \vec{\omega} \times \vec{\omega} \\ \left(\frac{d\vec{\omega}}{dt}\right)_{\text{fixed}} &= \left(\frac{d\vec{\omega}}{dt}\right)_{\text{rot}} \equiv \vec{\dot{\omega}}. \end{aligned} \quad (\text{X-57})$$

- viii) As such, the velocity of point P as measured in the fixed coordinate system is

$$\left(\frac{d\vec{r}'}{dt}\right)_{\text{fixed}} = \left(\frac{d\vec{R}}{dt}\right)_{\text{fixed}} + \left(\frac{d\vec{r}}{dt}\right)_{\text{fixed}}$$

$$\left(\frac{d\vec{r}'}{dt}\right)_{\text{fixed}} = \left(\frac{d\vec{R}}{dt}\right)_{\text{fixed}} + \left(\frac{d\vec{r}}{dt}\right)_{\text{rot}} + \vec{\omega} \times \vec{r}.$$

If we define

$$\vec{v}_f \equiv \vec{r}'_f \equiv \left(\frac{d\vec{r}'}{dt}\right)_{\text{fixed}} \quad (\text{X-58})$$

$$\vec{V} \equiv \vec{R}_f \equiv \left(\frac{d\vec{R}}{dt}\right)_{\text{fixed}} \quad (\text{X-59})$$

$$\vec{v}_r \equiv \vec{r}_r \equiv \left(\frac{d\vec{r}}{dt}\right)_{\text{rot}} \quad (\text{X-60})$$

we may write

$$\boxed{\vec{v}_f = \vec{V} + \vec{v}_r + \vec{\omega} \times \vec{r}}, \quad (\text{X-61})$$

where

\vec{v}_f = velocity relative to the fixed axes

\vec{V} = linear velocity of the moving origin

\vec{v}_r = velocity relative to the rotating axes

$\vec{\omega}$ = angular velocity of the rotating axes

$\vec{\omega} \times \vec{r}$ = velocity due to the rotation of the moving axes.

b) The Coriolis Force.

- i) Newton's 2nd law $\vec{F} = m\vec{a}$ is only valid in an inertial frame, therefore

$$\vec{F} = m\vec{a}_f = m \left(\frac{d\vec{v}_f}{dt}\right)_{\text{fixed}}, \quad (\text{X-62})$$

where the differentiation must be carried out with respect to the fixed system.

- ii) If we limit ourselves to cases of *constant angular acceleration* ($\dot{\omega} = 0$), using Eq. (X-61) we can write

$$\vec{F} = m\vec{R}_f + m \left(\frac{d\vec{v}_r}{dt}\right)_{\text{fixed}} + m\vec{\omega} \times \left(\frac{d\vec{r}}{dt}\right)_{\text{fixed}}. \quad (\text{X-63})$$

- iii) The second term can be evaluated by substituting \vec{v}_r for \vec{Q} in Eq. (X-56):

$$\begin{aligned} \left(\frac{d\vec{v}_r}{dt}\right)_{\text{fixed}} &= \left(\frac{d\vec{v}_r}{dt}\right)_{\text{rot}} + \vec{\omega} \times \vec{v}_r \\ &= \vec{a}_r + \vec{\omega} \times \vec{v}_r, \end{aligned} \quad (\text{X-64})$$

where \vec{a}_r is the acceleration in the rotating coordinate system.

- iv) The last term in Eq. (X-63) can be obtained directly from Eq. (X-55):

$$\begin{aligned} \vec{\omega} \times \left(\frac{d\vec{r}}{dt}\right)_{\text{fixed}} &= \vec{\omega} \times \left(\frac{d\vec{r}}{dt}\right)_{\text{rot}} + \vec{\omega} \times (\vec{\omega} \times \vec{r}) \\ &= \vec{\omega} \times \vec{v}_r + \vec{\omega} \times (\vec{\omega} \times \vec{r}). \end{aligned} \quad (\text{X-65})$$

- v) Combining Eqs. (X-63)-(X-65), we obtain

$$\vec{F} = m\vec{R}_f + m\vec{a}_r + 2m\vec{\omega} \times \vec{v}_r + m\vec{\omega} \times (\vec{\omega} \times \vec{r}), \quad \vec{\omega} = 0 \quad (\text{X-66})$$

where \vec{R}_f is the acceleration of the origin of the moving coordinate system relative to the fixed system.

- vi) In the case of trajectories on the Earth's surface, the origin of the rotating coordinate system is stationary (in the \vec{r} direction) with respect to the fixed coordinate system. As such, $\vec{R}_f = 0$ we can finally write

$$\vec{F} = m\vec{a}_f = m\vec{a}_r + m\vec{\omega} \times (\vec{\omega} \times \vec{r}) + 2m\vec{\omega} \times \vec{v}_r. \quad (\text{X-67})$$

- vii) From this equation, the effective force on a particle measured by an observer in the rotating frame

is then

$$\boxed{\vec{F}_{\text{eff}} = m\vec{a}_r = m\vec{a}_f - m\vec{\omega} \times (\vec{\omega} \times \vec{r}) - 2m\vec{\omega} \times \vec{v}_r}. \quad (\text{X-68})$$

which is what we wrote down in the Introduction of this section of the notes in Eq. (X-29).

**c) Coding Problems with $h \sim R_{\oplus}$ and $x \sim R_{\oplus}$ —
Part 1: The Fixed Coordinate System.**

i) Unlike the preceding case where $h \ll R_{\oplus}$ and $x \ll R_{\oplus}$, for our current problem we must use a three-dimensional coordinate system.

ii) In reality, the Earth is constantly being subjected to a variety of motions:

⇒ The Earth's own rotation.

⇒ The Earth's orbital velocity around the Sun.

⇒ The Sun's orbital velocity about the center of the Milky Way Galaxy.

⇒ The Milky Way's and Andromeda (M31) galaxy's motion towards each other.

⇒ The Local Group of galaxy's motion in the Virgo Supercluster of galaxies.

⇒ The Hubble flow of all of the galaxies, clusters, and superclusters in the Universe.

As can be seen, quite a lot of velocities to worry about. However, for our problem here, the Earth's rotational velocity will dominate these other velocities.

iii) Choose the fixed coordinate system's origin to lie at the Earth's center and its \hat{z}' direction to lie from a line from the center to the North Pole (follow-

ing the right-hand rule since the Earth is rotating counterclockwise as viewed from the North Pole) $\implies \vec{\omega} = \omega \hat{z}' = (7.272205 \times 10^{-5} \text{ rad/s}) \hat{z}'$.

iv) We will use spherical coordinates to describe our fixed coordinate system, however, instead of θ' which is measured from the z' axis, we will use the **latitude** angle $= \lambda' = 90^\circ - \theta'$ which is measured from the x' - y' plane \implies the **equatorial plane**. Hence, $\sin \theta' = \cos \lambda'$ and $\cos \theta' = \sin \lambda'$.

v) The \vec{r}' vector is measured with respect to the Earth's center ($= R_\oplus + z$, where z will be the height in the rotating frame with respect to sea level — see below). Since the Earth is an oblate spheroid, an accurate radius for the Earth is obtained with the equation

$$R_\oplus = \frac{R_{eq}(\oplus)}{e |\sin \lambda| + 1}, \quad (\text{X-69})$$

where $R_{eq}(\oplus) = 6.37853 \times 10^6 \text{ m}$ is the equatorial radius of the Earth (at sea level), λ is the latitude angle, and $e = 0.003393$ is the obliquity of the Earth.

vi) To define a reference point from which ϕ' is measured (*i.e.*, the x' axis — see Figure X-2), the background stars are used as the reference and the direction of the x' axis is from the center of the Earth to the **vernal equinox** on the celestial sphere. The vernal equinox is the intersection point of the Sun's path on the sky (*i.e.*, the **ecliptic**), which is the Earth's orbit projected onto the sky, and the

celestial equator, which is the Earth's equatorial plane projected on the sky.

$\implies \phi'$ completes one revolution in one **sidereal day** (= 24 *sidereal* hours exactly).

vii) Whereas the Earth's coordinate system is based on **latitude** (lines parallel to the equator) and **longitude** (lines \perp to latitude that run from the North to South Poles), both measured in degrees, the sky's coordinates are **declination** (DEC, like latitude and measured in degrees) and **right ascension** (RA, like longitude, but measured in units of time).

viii) *Local* sidereal time is determined by the current RA directly on the the local celestial meridian \implies hence 0^{hr} sidereal time occurs when the vernal equinox is on the local meridian.

ix) The time used in the changing ϕ' calculation is based on **Universal Time** $\implies 0^{\text{hr}}$ UT occurs when the vernal equinox is on the **celestial meridian** of the **Prime Meridian** in Greenwich, England which marks 0° longitude. The celestial meridian is an imaginary line on the sky that connects the north point on the horizon, the zenith (point directly overhead), and the south point on the horizon – it separates the east side of the sky from the west side of the sky. As such, Universal Time is the local sidereal time at the Prime Meridian.

- x) The Earth is subjected to extremely small, but unpredictable, variations in its rotation rate (mainly due to gravitational perturbations from other bodies in the Solar System and a non-smooth slowing down due to the tides raised from the Sun and Moon). To precisely predict the positions of bodies in the Solar System, we require a steady time standard.

⇒ Universal Time is then replaced by **Ephemeris Time** (E.T.) in celestial mechanics.

⇒ At the beginning of 1900 A.D., an **ephemeris second** was defined as $1/31,556,925.97474$ the length of the tropical year 1900 and both U.T. and E.T. were in agreement.

⇒ Today, these times differ by about 56 seconds.

- xi) However for this course, we will just use U.T. To calculate ϕ' , use the following formula:

$$\phi' = 2\pi \frac{t_{\text{U.T.}}}{24 \text{ hr}}, \quad (\text{X-70})$$

where $t_{\text{U.T.}}$ is the current Universal Time in decimal sidereal hours.

- xii) As such, your code should have the ability for the user to input:

⇒ **A time of launch (in U.T. sidereal hours).**

⇒ **The latitude of launch (in decimal degrees).**

⇒ **The altitude of the ground with respect to sea level.**

d) **Coding Problems with $h \sim R_{\oplus}$ and $x \sim R_{\oplus}$ —
Part 2: The Rotating Coordinate System.**

i) We now have all of the fixed (*primed*) frame coordinates (*i.e.*, r' , λ' , and ϕ'), we now need the rotating frame coordinates.

$\implies x$ is defined in the **eastern** direction.

$\implies y$ is defined in the **northern** direction.

$\implies z$ is the altitude (which follows from the right-hand rule).

ii) Rotational coordinate frame transformations are then made with

$$r = (x^2 + y^2 + z^2)^{1/2} \quad (\text{X-71})$$

$$\theta = \cos^{-1} \left(\frac{z}{(x^2 + y^2 + z^2)^{1/2}} \right) \quad (\text{X-72})$$

$$\phi = \tan^{-1} \frac{y}{x}, \quad (\text{X-73})$$

where r is measured from the launch point (the rotating origin), θ is the angle from the z (*i.e.*, altitude) axis (note that this makes θ different from what it was in the $h \ll R_{\oplus}$ case when it was measured from the ground), and ϕ is the angle subtended from the east point on the horizon moving towards north.

iii) Now your code should allow the user to input:

⇒ **A launch velocity (typically in m/s)**
 $= \dot{r}$.

⇒ **The projection angle of launch (in decimal degrees) = γ .**

⇒ **The direction angle with respect to the east direction rotating towards north = ϕ .**

iv) Then from these in input, r will be determined in the next step and $\theta = 90^\circ - \gamma$.

e) **Coding Problems with $h \sim R_\oplus$ and $x \sim R_\oplus$ —
 Part 3: Solving the System of Equations.**

i) Calculate the Coriolis and centrifugal cross products in Eq. (X-68).

ii) Solve for v_r in Eq. (X-61) by using the forward-difference technique shown in Eqs. (X-43). However, now, replace y with z and the two x equations become four — 2 for x and 2 for y which will now have the same functional form. Also, instead of g , now use a_r from Eq. (X-68) in the z direction where $a_f = g$ in this equation.

iii) Solve for the unknowns: downrange distance, maximum height, impact velocity, etc.

C. The Theory of Orbits.

1. Reduced Mass.

a) For two masses m_1 and m_2 , Newton's Law of Gravity is described as a force between two particles separated by a distance \vec{r} . However, if the origin does not lie along that line, then $\vec{r} \equiv \vec{r}_1 - \vec{r}_2$.

b) The Lagrangian for such a system may be written as

$$\mathcal{L} = \frac{1}{2}m_1|\dot{r}_1|^2 + \frac{1}{2}m_2|\dot{r}_2|^2 - U(r) . \quad (\text{X-74})$$

c) For two bodies in orbit about each other, it is convenient to select the origin such that it lies on the **center of mass** along \vec{r} . The center of mass is defined by the equation:

$$m_1\vec{r}_1 + m_2\vec{r}_2 = 0 . \quad (\text{X-75})$$

d) This equation, combined with $\vec{r} = \vec{r}_1 - \vec{r}_2$, yields

$$\left. \begin{aligned} \vec{r}_1 &= \frac{m_2}{m_1+m_2} \vec{r} \\ \vec{r}_2 &= -\frac{m_1}{m_1+m_2} \vec{r} \end{aligned} \right\} . \quad (\text{X-76})$$

e) Substituting Eq. (X-76) into Eq. (X-74) gives

$$\boxed{\mathcal{L} = \frac{1}{2}\mu|\dot{\vec{r}}|^2 - U(r)} , \quad (\text{X-77})$$

where μ is the **reduced mass**,

$$\mu \equiv \frac{m_1 m_2}{m_1 + m_2} = \frac{m_1 m_2}{M} , \quad (\text{X-78})$$

and $M = m_1 + m_2$ is the total mass of the system.

f) By introducing the reduced mass, we have reduced the motion of two bodies to an *equivalent one-body problem* in which we must determine only the motion of a “particle” of mass μ in the central potential field described by $U(r)$.

2. Conservation Theorems — First Integrals of the Motion.

- a) In a spherically symmetric force field, angular momentum is conserved:

$$\vec{L} = \vec{r} \times \vec{p} = m\vec{r} \times \dot{\vec{r}} = \text{constant}, \quad (\text{X-79})$$

where \vec{p} is the linear momentum.

- b) From this conservation law, we see that the radius vector and the linear velocity are coplanar and \perp to the constant angular momentum vector.

- i) This means that the $\dot{\phi}$ term is zero since \vec{L} lies in the direction of \hat{z} from the angular momentum equation above.

- ii) As such, we can write \vec{r} in Eq. (X-77) using only plane polar coordinates:

$$\mathcal{L} = \frac{1}{2}\mu (\dot{r}^2 + r^2\dot{\theta}^2) - U(r) . \quad (\text{X-80})$$

- c) Since the Lagrangian is cyclic in θ , the angular momentum conjugate to the coordinate θ is conserved:

$$\dot{p}_\theta = \frac{\partial \mathcal{L}}{\partial \theta} = 0 = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\theta}} \quad (\text{X-81})$$

or

$$p_\theta \equiv \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = \mu r^2 \dot{\theta} = \text{constant}. \quad (\text{X-82})$$

- d) The quantity p_θ is a *first integral* of the motion. From this point forward, we will denote its constant value by the symbol ℓ :

$$\ell \equiv \mu r^2 \dot{\theta} = \text{constant}. \quad (\text{X-83})$$

Figure X-5: Geometry setup of Kepler's 2nd Law of Motion.

- e) Referring to Fig. (X-5), we see that describing the path $\vec{r}(t)$, the radius vector sweeps out an area $\frac{1}{2}r^2 d\theta$ in a time interval dt :

$$dA = \frac{1}{2}r^2 d\theta , \quad (\text{X-84})$$

and dividing by the time interval, the *areal velocity* is

$$\begin{aligned} \frac{dA}{dt} &= \frac{1}{2}r^2 \frac{d\theta}{dt} = \frac{1}{2}r^2\dot{\theta} \\ &= \frac{\ell}{2\mu} = \text{constant}, \end{aligned} \quad (\text{X-85})$$

which shows that the areal velocity is constant in time
 \implies **Kepler's Second Law of Planetary Motion.**

- f) The conservation of energy gives us

$$E = T + U = \text{constant} \quad (\text{X-86})$$

$$= \frac{1}{2}\mu (\dot{r}^2 + r^2\dot{\theta}^2) + U(r) , \quad (\text{X-87})$$

or

$$\boxed{E = \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\frac{\ell^2}{\mu r^2} + U(r)} . \quad (\text{X-88})$$

3. Equation of Motion.

- a) Using the total energy equation (Eq. X-88), we can solve for the velocity:

$$\dot{r} = \frac{dr}{dt} = \sqrt{\frac{2}{\mu}(E - U) - \frac{\ell^2}{\mu^2 r^2}} . \quad (\text{X-89})$$

- b) Note that

$$d\theta = \frac{d\theta}{dt} \frac{dt}{dr} dr = \frac{\dot{\theta}}{\dot{r}} dr = \frac{\ell/\mu r^2}{\dot{r}} dr , \quad (\text{X-90})$$

or

$$\dot{r} = \frac{\ell}{\mu r^2} \frac{dr}{d\theta} , \quad (\text{X-91})$$

hence

$$\frac{\ell}{\mu r^2} \frac{dr}{d\theta} = \sqrt{\frac{2}{\mu}(E - U) - \frac{\ell^2}{\mu^2 r^2}} . \quad (\text{X-92})$$

- c) This leads to an equation of $\theta(r)$:

$$\theta(r) = \int \frac{(\ell/r^2) dr}{\sqrt{2\mu(E - U) - \ell^2/2\mu r^2}} + \text{constant}. \quad (\text{X-93})$$

- d) If $F(r) \propto r^n$, Eq. (X-93) becomes an *elliptical integral*.

- i) For $n = 1, -2$, and $-3 \rightarrow$ solutions are circular functions.

- ii) $n = 1$ gives harmonic motion.

- iii) $n = -2$ gives a central-force law (*e.g.*, gravity, see later).

- iv) $n = -3$ gives an equation that is not important in physics.

e) We can also solve this problem with Lagrangians:

$$\frac{\partial \mathcal{L}}{\partial r} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{r}} = 0, \quad (\text{X-94})$$

where the Lagrangian is given by Eq. (X-80).

i) Term one of Eq. (X-94) is

$$\frac{\partial \mathcal{L}}{\partial r} = \mu r \dot{\theta}^2 - \frac{\partial U}{\partial r}.$$

ii) Term two of Eq. (X-94) is

$$\frac{\partial \mathcal{L}}{\partial \dot{r}} = \mu \dot{r}, \quad \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{r}} = \mu \ddot{r}.$$

iii) Putting these two equations together gives

$$\mu r \dot{\theta}^2 - \mu \ddot{r} - \frac{\partial U}{\partial r} = 0$$

or

$$\mu(\ddot{r} - r\dot{\theta}^2) = -\frac{\partial U}{\partial r} = F(r). \quad (\text{X-95})$$

f) Now let

$$u \equiv \frac{1}{r}, \quad (\text{X-96})$$

then

$$\frac{du}{d\theta} = -\frac{1}{r^2} \frac{dr}{d\theta} = -\frac{1}{r^2} \frac{dr}{dt} \frac{dt}{d\theta} = -\frac{1}{r^2} \frac{\dot{r}}{\dot{\theta}}. \quad (\text{X-97})$$

g) Since $\dot{\theta} = \ell/\mu r^2$, we get

$$\frac{du}{d\theta} = -\frac{\mu}{\ell} \dot{r}. \quad (\text{X-98})$$

h) Next,

$$\begin{aligned} \frac{d}{d\theta} \frac{du}{d\theta} &= \frac{d^2 u}{d\theta^2} = \frac{d}{d\theta} \left(-\frac{\mu}{\ell} \dot{r} \right) = \frac{dt}{d\theta} \frac{d}{dt} \left(-\frac{\mu}{\ell} \dot{r} \right) \\ &= -\frac{\mu}{\ell \dot{\theta}} \ddot{r} = -\frac{\mu^2}{\ell^2} r^2 \ddot{r}. \end{aligned} \quad (\text{X-99})$$

i) Therefore since $u = 1/r$,

$$\ddot{r} = -\frac{\ell^2}{\mu^2} u^2 \frac{d^2 u}{d\theta^2} \quad (\text{X-100})$$

$$r\dot{\theta}^2 = \frac{\ell^2}{\mu^2} u^3. \quad (\text{X-101})$$

j) Using this in Eq. (X-95) gives

$$\frac{d^2 u}{d\theta^2} + u = -\frac{\mu}{\ell^2} \frac{1}{u^2} F(u), \quad (\text{X-102})$$

or

$$\boxed{\frac{d^2}{d\theta^2} \left(\frac{1}{r} \right) + \frac{1}{r} = -\frac{\mu r^2}{\ell^2} F(r)} \quad (\text{X-103})$$

\implies this equation is useful if we wish to find $r = r(\theta)$.

4. Orbits in a Central Field.

a) From Eq. (X-89), the velocity vanishes when

$$E - U(r) - \frac{\ell^2}{2\mu r^2} = 0. \quad (\text{X-104})$$

i) These are *turning points* in the motion.

ii) Two roots are obtained from Eq. (X-104): r_{\min}
and r_{\max} such that $r_{\min} \leq r \leq r_{\max}$.

b) Certain combinations of $U(r)$, E , and ℓ will produce a single root for Eq. (X-104), then $\dot{r} = 0$ for all times, hence $r = \text{constant} \rightarrow$ the orbit is *circular*.

c) If the motion of the particle in $U(r)$ is periodic \rightarrow orbit is **closed**.

d) If the orbit does not close on itself (*i.e.*, does not repeat itself) \rightarrow orbit is **open**.

- e) Orbital motion is symmetric in time between r_{\min} and r_{\max} , hence $\Delta\theta = 2\theta(r)$ or

$$\Delta\theta(r) = 2 \int_{r_{\min}}^{r_{\max}} \frac{(\ell/r^2) dr}{\sqrt{2\mu(E - U - \ell^2/2\mu r^2)}}. \quad (\text{X-105})$$

- i) The path will be closed only if $\Delta\theta$ is a rational function of 2π ($\Delta\theta = 2\pi \cdot \frac{a}{b}$, where ‘ a ’ and ‘ b ’ are integers).
- ii) After ‘ b ’ periods, r will have made ‘ a ’ complete revolutions and return to its original position.
- iii) If $U(r) \propto r^{n+1}$, then closed non-circular paths can result only if $n = -2$ (inverse-square law) or $+1$ (harmonic oscillator).

5. Centrifugal Energy and Effective Potential.

- a) Note that $\left[\frac{\ell^2}{2\mu r^2}\right] = [E]$ (see Eq. X-105) and

$$\frac{\ell^2}{2\mu r^2} = \frac{1}{2}\mu r^2 \dot{\theta}^2. \quad (\text{X-106})$$

Now let

$$U_c \equiv \frac{\ell^2}{2\mu r^2} \quad (\text{X-107})$$

and

$$F_c = -\frac{\partial U_c}{\partial r} = \frac{\ell^2}{\mu r^3} = \mu r \dot{\theta}^2 = \begin{array}{c} \text{centrifugal} \\ \text{force} \end{array} \quad (\text{X-108})$$

(although not a ‘force’ in the ordinary sense of the word).

- b) Hence, let $U_c \equiv$ **centrifugal potential energy**.
- c) Then, the **effective potential energy** is

$$V(r) = U(r) + \frac{\ell^2}{2\mu r^2}. \quad (\text{X-109})$$

d) For gravity,

$$F(r) = -\frac{Gm_1m_2}{r^2}, \quad (\text{X-110})$$

so

$$U(r) = -\int F(r) dr = -\frac{Gm_1m_2}{r} \quad (\text{X-111})$$

and

$$V(r) = -\frac{Gm_1m_2}{r} + \frac{\ell^2}{2\mu r^2}, \quad (\text{X-112})$$

where $V \rightarrow 0$ as $r \rightarrow \infty$. See Fig. (X-6) on the next page for plots of these potentials.

- i) For the bottom plot in Fig. (X-6), if $E = E_1 > 0$, the motion is unbounded.
- ii) If $0 > E(= E_2) > E_3$, the motion has turning points (*i.e.*, bounded between r_2 and $r_4 \rightarrow$ the *apsidal* distances).
- iii) If $E = E_3$, only one solution exists \rightarrow motion is circular.
- iv) Note that $E < E_3$ is not possible $\implies \dot{r}$ would be imaginary!

6. Planetary Motion — Kepler's Problem.

a) Let us re-examine Eq. (X-93).

- i) For the potential energy given by Eq. (X-111) let's define $k = Gm_1m_2$ (typically we will let m_1 be the bigger mass such that $m_1 \gg m_2$) giving the potential energy as

$$U(r) = -\frac{Gm_1m_2}{r} = -\frac{k}{r}. \quad (\text{X-113})$$

Figure X-6: The various potentials in a gravitational field.

- ii) If we choose the origin of θ so that the integration constant in Eq. (X-93) is zero, we can show (with a little algebra) that

$$\cos \theta = \frac{\frac{\ell^2}{\mu k} \cdot \frac{1}{r} - 1}{\sqrt{1 + \frac{2E\ell^2}{\mu k^2}}} . \quad (\text{X-114})$$

- iii) Let us now define the following constants:

$$\alpha \equiv \frac{\ell^2}{\mu k} \quad (\text{X-115})$$

$$\varepsilon \equiv \sqrt{1 + \frac{2E\ell^2}{\mu k^2}} . \quad (\text{X-116})$$

- b) Then we can rewrite Eq. (X-114) as

$$\boxed{\frac{\alpha}{r} = 1 + \varepsilon \cos \theta} . \quad (\text{X-117})$$

- i) This is the equation of a conic section with one focus at the origin.
- ii) The quantity ε is called the **eccentricity** of the orbit.
- iii) The quantity 2α is termed the **latus rectum** of the orbit.
- c) The minimum value for r occurs when $\cos \theta$ is a maximum (*i.e.*, for $\theta = 0$). Thus the choice of the integration constant in Eq. (X-93) be zero corresponds to measuring θ from r_{\min} .
- i) For an arbitrary orbit, r_{\min} is called the **pericenter**.
- ii) For solar orbits it is called the **perihelion**.

- iii) For objects in orbit about the Earth, it is called the **perigee**.
- d) The maximum value for r occurs when $\cos \theta$ is a minimum (*i.e.*, for $\theta = 180^\circ$).
- i) For an arbitrary orbit, r_{\max} is called the **apocenter**.
- ii) For solar orbits it is called the **aphelion**.
- iii) For objects in orbit about the Earth, it is called the **apogee**.
- e) Various values of the eccentricity (and, hence, of the energy) classify the orbits according to the different conic sections (see Figure X-7):

$\varepsilon > 1$	$E > 0$	(hyperbola)
$\varepsilon = 1$	$E = 0$	(parabola)
$0 < \varepsilon < 1$	$U_{\min} < E < 0$	(ellipse)
$\varepsilon = 0$	$E = U_{\min}$	(circle)
$\varepsilon < 0$	$E < U_{\min}$	(not allowed)

- f) For the case of planetary motion, the orbits are ellipses as per **Kepler's First Law of Planetary Motion** (see Figure X-8).
- i) The **semimajor axis** of an elliptical orbit is given by

$$a = \frac{\alpha}{1 - \varepsilon^2} = \frac{k}{2|E|} = \frac{Gm_1m_2}{2|E|}, \quad (\text{X-118})$$

as such, the semimajor axis depends only on the energy of the particle, or vice versa,

$$|E| = \frac{k}{2a} = \frac{Gm_1m_2}{2a}. \quad (\text{X-119})$$

Figure X-7: Possible orbits in a $1/r$ potential follow the conic sections shown here.

Figure X-8: Details of the elliptical orbit.

- ii) The **semiminor axis** of an elliptical orbit is given by

$$b = \frac{\alpha}{\sqrt{1 - \varepsilon^2}} = \frac{\ell}{2\mu|E|} , \quad (\text{X-120})$$

as such, the semiminor axis depends both on E and ℓ . Comparing Eq. (X-118) with Eq. (X-120), we can write the semiminor axis in terms of the semimajor as

$$b = a\sqrt{1 - \varepsilon^2} = \sqrt{\alpha a} . \quad (\text{X-121})$$

- g) The extrema (*i.e.*, turning points) of the orbit is given by

$$r_{\min} = a(1 - \varepsilon) = \frac{\alpha}{1 + \varepsilon} \quad (\text{X-122})$$

$$r_{\max} = a(1 + \varepsilon) = \frac{\alpha}{1 - \varepsilon} \quad (\text{X-123})$$

- h) To find the period of elliptical motion, take the areal velocity equation (Eq. X-85) and integrate:

$$\begin{aligned} dt &= \frac{2\mu}{\ell} dA \\ \int_0^\tau dt &= \frac{2\mu}{\ell} \int_0^A dA \\ \tau &= \frac{2\mu}{\ell} A , \end{aligned} \quad (\text{X-124})$$

where the entire area of the ellipse A is swept out in one orbital period τ .

- i) Since $A = \pi ab$ for an ellipse, we can write the orbital period in terms of the energy of a particle in orbit a

$$\begin{aligned} \tau &= \frac{2\mu}{\ell} \pi ab = \frac{2\mu}{\ell} \frac{k}{2|E|} \frac{\ell}{\sqrt{2\mu|E|}} \\ &= \pi k \sqrt{\frac{\mu}{2}} |E|^{-3/2} . \end{aligned} \quad (\text{X-125})$$

j) Likewise, we can use Eqs. (X-115) and (X-121) to write

$$\tau = \frac{2\mu}{\ell} \pi ab = \frac{2\mu}{\ell} \pi a \sqrt{\alpha a} = \frac{2\pi\mu}{\ell} \sqrt{\frac{\ell^2}{\mu k}} a^{3/2}, \quad (\text{X-126})$$

or

$$\boxed{\tau^2 = \frac{4\pi^2\mu}{k} a^3 = \frac{4\pi^2}{G(m_1 + m_2)} a^3}, \quad (\text{X-127})$$

which is the analytic proof to **Kepler's Third Law of Planetary Motion**. Note that for planets in the solar system, $m_1 = M_\odot \gg m_2$ (where m_2 is the mass of a planet), giving

$$\tau^2 \cong \frac{4\pi^2}{GM_\odot} a^3 = K_\odot a^3,$$

where K_\odot is a constant (as long as $m_2 \ll M_\odot$) \implies this is the 3rd law form as Kepler stated it.

7. Kepler's Equation.

a) We have written an equation of $r(\theta)$ (see Eq. X-117). We will now develop an equation for θ , called the **true anomaly**, as a function of time.

b) We will start by setting up the ratio

$$\frac{A}{\tau} = \frac{\pi ab}{\tau} = \frac{dA}{dt}. \quad (\text{X-128})$$

i) Let $\theta = 0$ at $t = 0$, where $dA = \frac{1}{2}r^2d\theta$, then

$$\frac{\pi ab}{\tau} \int_0^t dt' = \int_0^A dA' = \frac{1}{2} \int_0^\theta r^2 d\theta. \quad (\text{X-129})$$

ii) Using Eq. (X-117), we can write r in terms of θ as

$$r = \frac{\alpha}{1 + \varepsilon \cos \theta}, \quad (\text{X-130})$$

and hence

$$\begin{aligned} \frac{\pi ab}{\tau} t &= \frac{\alpha^2}{2} \int_0^\theta \frac{d\theta}{(1 + \varepsilon \cos \theta)^2} \\ &= \frac{\alpha^2}{2(1 - \varepsilon^2)} \left[\frac{2}{\sqrt{1 - \varepsilon^2}} \tan^{-1} \left(\frac{(1 - \varepsilon) \tan(\theta/2)}{\sqrt{1 - \varepsilon^2}} \right) \right. \\ &\quad \left. - \frac{\varepsilon \sin \theta}{1 + \varepsilon \cos \theta} \right]. \end{aligned}$$

iii) Note that

$$ab = \alpha^2 (1 - \varepsilon^2)^{-3/2}, \quad (\text{X-131})$$

so we can write

$$\begin{aligned} \frac{2\pi t}{\tau} &= 2 \tan^{-1} \left[\sqrt{\frac{1 - \varepsilon}{1 + \varepsilon}} \tan \frac{\theta}{2} \right] \\ &\quad - \frac{\varepsilon \sqrt{1 - \varepsilon^2} \sin \theta}{1 + \varepsilon \cos \theta}. \quad (\text{X-132}) \end{aligned}$$

c) As can be seen, retrieving $\theta(t)$ from Eq. (X-132) will not be easy! Due to this difficulty, we will approach the problem through analytic geometry.

i) We will start by circumscribing the ellipse (representing an orbit) with a circle and define a coordinate system with the origin set at one of the foci of the ellipse as shown in Figure X-9.

ii) The equation for the ellipse is

$$\frac{(x + a\varepsilon)^2}{a^2} + \frac{y^2}{b^2} = 1. \quad (\text{X-133})$$

- The circle has radius of ‘ a .’
- Point ‘ P ’ lies on the ellipse, ‘ Q ’ on the circle.

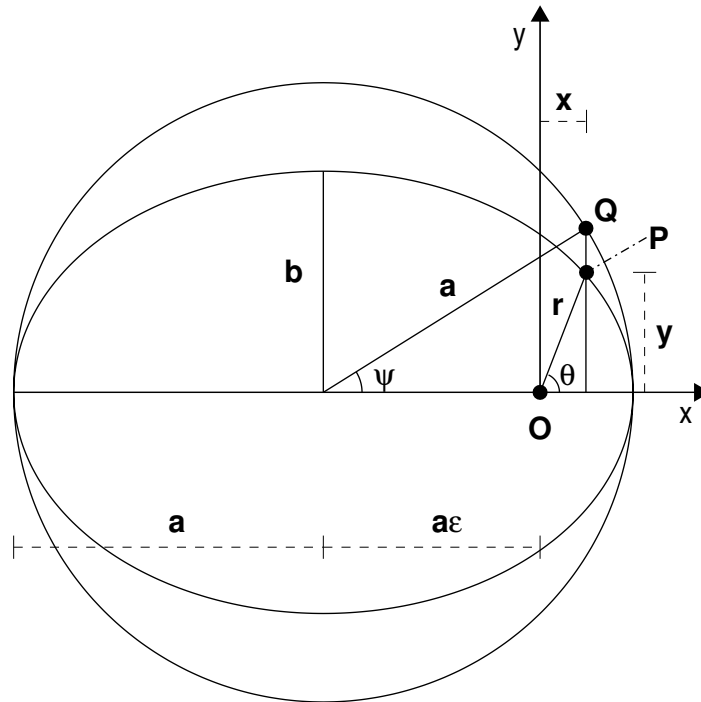


Figure X-9: The geometry used in the derivation of Kepler's Equation. Note that the Sun is located at one of the foci labeled as the origin 'O'.

- d)** We now define a new angle ψ (called the **eccentric anomaly**) as the angle between the circle's radius vector and the x axis. In relation to Eq. (X-133), the following definitions are made for the trigonometric functions of the eccentric anomaly:

$$\cos \psi \equiv \frac{x + a\varepsilon}{a} \quad (\text{X-134})$$

$$\sin \psi \equiv \frac{y}{b}. \quad (\text{X-135})$$

Investigating Figure X-9, it is easy to see from trigonometry how the cosine equation (Eq. X-134) arises. However the sine equation is not as easily justified. Unfortunately, I have been unable to prove Eq. (X-135) with geometry or unable to find a reference that describes the proof for this equation. As such, we will just take it as an assumption. A little algebra can be used to derive:

$$x = a(\cos \psi - \varepsilon) \quad (\text{X-136})$$

$$y = b \sin \psi = a\sqrt{1 - \varepsilon^2} \sin \psi. \quad (\text{X-137})$$

- e) Now looking at the smaller θ triangle in Figure (X-9), we use the equations above in the Pythagorean theorem and carry out a bit of algebra to derive:

$$\begin{aligned}
 r^2 &= x^2 + y^2 = a^2 (\cos \psi - \varepsilon)^2 + a^2 (1 - \varepsilon^2) \sin^2 \psi \\
 &= a^2 (\cos^2 \psi - 2\varepsilon \cos \psi + \varepsilon^2) + a^2 \sin^2 \psi - a^2 \varepsilon^2 \sin^2 \psi \\
 &= a^2 \cos^2 \psi - 2a^2 \varepsilon \cos \psi + a^2 \varepsilon^2 + a^2 \sin^2 \psi - a^2 \varepsilon^2 \sin^2 \psi \\
 &= a^2 (\cos^2 \psi + \sin^2 \psi) + a^2 \varepsilon^2 (1 - \sin^2 \psi) - 2a^2 \varepsilon \cos \psi \\
 &= a^2 + a^2 \varepsilon^2 \cos^2 \psi - 2a^2 \varepsilon \cos \psi \\
 &= a^2 (1 - 2\varepsilon \cos \psi + \varepsilon^2 \cos^2 \psi) \\
 &= a^2 (1 - \varepsilon \cos \psi)^2, \quad \text{or} \\
 r &= a(1 - \varepsilon \cos \psi). \tag{X-138}
 \end{aligned}$$

- f) Now rewrite Eq. (X-130) as

$$\varepsilon r \cos \theta = a(1 - \varepsilon^2) - r, \tag{X-139}$$

followed by adding εr to both sides which gives

$$\varepsilon r(1 + \cos \theta) = (1 - \varepsilon)[a(1 + \varepsilon) - r]. \tag{X-140}$$

- g) Substitute Eq. (X-138) for the last term in this equation giving

$$\varepsilon r(1 + \cos \theta) = (1 - \varepsilon)[a(1 + \varepsilon) - a(1 - \varepsilon \cos \psi)]$$

or

$$r(1 + \cos \theta) = a(1 - \varepsilon)(1 + \cos \psi). \tag{X-141}$$

- h) Subtracting εr from both sides of Eq. (X-139) [after using Eq. (X-138), as we did before] results in

$$r(1 - \cos \theta) = a(1 + \varepsilon)(1 - \cos \psi). \tag{X-142}$$

- i) Dividing Eq. (X-142) by Eq. (X-141) gives:

$$\frac{1 - \cos \theta}{1 + \cos \theta} = \frac{1 + \varepsilon}{1 - \varepsilon} \cdot \frac{1 - \cos \psi}{1 + \cos \psi}. \tag{X-143}$$

j) If we make use of the trigonometric identity

$$\tan \frac{\alpha}{2} = \sqrt{\frac{1 - \cos \alpha}{1 + \cos \alpha}} ,$$

we can take the square root of Eq. (X-143) to write

$$\tan \frac{\theta}{2} = \sqrt{\frac{1 + \varepsilon}{1 - \varepsilon}} \tan \frac{\psi}{2} . \quad (\text{X-144})$$

k) Now, $\theta(t)$ can be found directly from $\psi(t)$.

i) Differentiating Eq. (X-144) gives

$$d\theta = \sqrt{\frac{1 + \varepsilon}{1 - \varepsilon}} \frac{\cos^2(\theta/2)}{\cos^2(\psi/2)} d\psi . \quad (\text{X-145})$$

ii) Rewriting Eq. (X-141) gives

$$\begin{aligned} r &= a(1 - \varepsilon) \frac{1 + \cos \psi}{1 + \cos \theta} \\ &= a(1 - \varepsilon) \frac{\cos^2(\psi/2)}{\cos^2(\theta/2)} . \end{aligned} \quad (\text{X-146})$$

l) We will now make use of this expression for r in Eq. (X-129):

$$\frac{\pi ab}{\tau} t = \frac{1}{2} \int_0^\theta r^2 d\theta .$$

i) However for this integral, let $r^2 = (r_1)(r_2)$, where r_1 is given by Eq. (X-138) and r_2 is given by Eq. (X-146).

ii) Then, using Eq. (X-145) for $d\theta$ we get

$$\begin{aligned} r^2 d\theta &= [a(1 - \varepsilon \cos \psi)] \left[a(1 - \varepsilon) \frac{\cos^2(\psi/2)}{\cos^2(\theta/2)} \right] \\ &\quad \left[\sqrt{\frac{1 + \varepsilon}{1 - \varepsilon}} \cdot \frac{\cos^2(\theta/2)}{\cos^2(\psi/2)} d\psi \right] \\ &= a^2 \sqrt{1 - \varepsilon^2} (1 - \varepsilon \cos \psi) d\psi . \end{aligned}$$

iii) So

$$\frac{\pi ab}{\tau} t = \frac{a^2 \sqrt{1 - \varepsilon^2}}{2} \int_0^\psi (1 - \varepsilon \cos \psi) d\psi .$$

iv) Since $b = a\sqrt{1 - \varepsilon^2}$, $ab = a^2\sqrt{1 - \varepsilon^2}$, and

$$\frac{2\pi t}{\tau} = \psi - \varepsilon \sin \psi . \quad (\text{X-147})$$

v) Let $M = 2\pi t/\tau = \mathbf{mean\ anomaly} \implies$ measures angular deviation of a body moving in a circular orbit of period τ (note that $M = 2\pi$ when the mass completes one orbit, $t = \tau$), then

$$\boxed{M = \psi - \varepsilon \sin \psi ,} \quad (\text{X-148})$$

which is referred to as **Kepler's Equation**.

m) In order to find $\psi(t)$, Kepler's Equation must be inverted by some approximation procedure which usually involves expanding the sine term in a Taylor's series (see E.W. Brown 1931, *Monthly Notices of the Royal Astronomical Society*, **92**, 104). Then, Eq. (X-144) relates ψ and θ and hence the time dependence of the *true* anomaly (*i.e.*, the anomaly for an elliptical orbit) can be found.

8. Following the same geometric arguments that we made to derive Kepler's Equation, we can develop a velocity equation as a function of r . The orbital velocity can be found with

$$v^2 = \dot{x}^2 + \dot{y}^2 , \quad (\text{X-149})$$

or

$$\begin{aligned} v^2 &= a^2 \dot{\psi}^2 \sin^2 \psi + a^2 (1 - \varepsilon^2) \dot{\psi}^2 \cos^2 \psi \\ &= a^2 \dot{\psi}^2 (1 - \varepsilon^2 \cos^2 \psi) . \end{aligned} \quad (\text{X-150})$$

a) Differentiating Eq. (X-147) gives

$$\frac{2\pi}{\tau} = \dot{\psi}(1 - \varepsilon \cos \psi) . \quad (\text{X-151})$$

b) Solving this equation for $\dot{\psi}$ and inserting this value in Eq. (X-150) gives

$$\begin{aligned} v^2 &= \left(\frac{2\pi}{\tau}\right)^2 a^2 \frac{1 - \varepsilon^2 \cos^2 \psi}{(1 - \varepsilon \cos \psi)^2} \\ &= \left(\frac{2\pi}{\tau}\right)^2 a^2 \frac{1 + \varepsilon \cos \psi}{1 - \varepsilon \cos \psi} \\ &= \left(\frac{2\pi}{\tau}\right)^2 a^2 \frac{2 - (1 - \varepsilon \cos \psi)}{1 - \varepsilon \cos \psi} . \end{aligned} \quad (\text{X-152})$$

c) Note that

$$\frac{r}{a} = 1 - \varepsilon \cos \psi ,$$

so we can write

$$v^2 = \left(\frac{2\pi}{\tau}\right)^2 a^3 \left(\frac{2}{r} - \frac{1}{a}\right) . \quad (\text{X-153})$$

d) Using Kepler's 3rd Law:

$$v^2 = \frac{Gm_1m_2}{\mu} \left(\frac{2}{r} - \frac{1}{a}\right)$$

or

$$\boxed{v^2 = G(m_1 + m_2) \left(\frac{2}{r} - \frac{1}{a}\right)} . \quad (\text{X-154})$$

D. Programming Orbital Motion in the Solar System.

1. As we have seen, the analytic solution of planetary motion is extremely complicated. This problem is much easier to solve numerically.
2. Here, we will limit ourselves to motion of a mass ($m_2 = m_p = m$) in orbit about the Sun ($m_1 = M_\odot$). Using these masses in Eq.

Figure X-10: Coordinate system for describing the motion of a planet (or comet or asteroid) in orbit about the Sun. The Sun is at the origin and the planet is located at coordinate (x, y) .

(X-6) Newton's law of gravity becomes

$$\vec{F}_g = -\frac{GM_\odot m}{r^2} \hat{r}, \quad (\text{X-155})$$

once again, the negative sign indicates that this is an attractive force.

- a)** As we did for trajectories (see Page X-11), we will break this force up into an x component and a y component as shown in Figure X-10. Then, using Newton's 2nd law we have

$$\frac{d^2x}{dt^2} = \frac{F_{g,x}}{m} \quad (\text{X-156})$$

$$\frac{d^2y}{dt^2} = \frac{F_{g,y}}{m}, \quad (\text{X-157})$$

where $F_{g,x}$ and $F_{g,y}$ are the x and y components of the gravitational force.

- b)** It is standard convention to define the x axis along the major axis of the ellipse and the y axis parallel to the

minor axis with its origin centered at the primary mass location.

c) From Figure X-10 we have

$$F_{g,x} = -\frac{GM_{\odot}m}{r^2} \cos \theta = -\frac{GM_{\odot}mx}{r^3} \quad (\text{X-158})$$

and

$$F_{g,y} = -\frac{GM_{\odot}m}{r^2} \sin \theta = -\frac{GM_{\odot}my}{r^3} . \quad (\text{X-159})$$

d) As can be seen by the equations above, when x is positive, the force in the x direction will drive the orbiting mass towards the y axis, and $y > 0$, $F_{g,y}$ towards x axis. When $x < 0$, $F_{g,x} > 0$ which will force the mass back towards the y axis with a similar effect for $F_{g,y}$ forcing the mass back towards the x axis, hence keeping the mass in orbit about the primary mass.

3. We will break the second-order differential equations given in Eqs. (X-156) and (X-157) into a pair of two first-order differential equations:

$$\frac{dv_x}{dt} = -\frac{GM_{\odot}x}{r^3} \quad (\text{X-160})$$

$$\frac{dx}{dt} = v_x \quad (\text{X-161})$$

$$\frac{dv_y}{dt} = -\frac{GM_{\odot}y}{r^3} \quad (\text{X-162})$$

$$\frac{dy}{dt} = v_y . \quad (\text{X-163})$$

4. Next, we convert these differential equation into difference equations using a forward-difference scheme:

$$v_{x,i+1} = v_{x,i} - \frac{GM_{\odot}x_i}{r_i^3} \Delta t \quad (\text{X-164})$$

$$x_{i+1} = x_i + v_{x,i+1} \Delta t \quad (\text{X-165})$$

$$v_{y,i+1} = v_{y,i} - \frac{GM_{\odot}y_i}{r_i^3} \Delta t \quad (\text{X-166})$$

$$y_{i+1} = y_i + v_{y,i+1} \Delta t . \quad (\text{X-167})$$

5. To carry out an orbit calculation, one must first define the semi-major axis size ‘ a ’ and the eccentricity ε of the orbit as input parameters. In addition, the masses of the primary (*e.g.*, the Sun for solar orbits) and of the secondary must be supplied.
6. Finally, we need to set the initial values for x , y , v_x , and v_y , and the initial time interval Δt . This is usually done by starting at the pericenter point with $\theta = \theta_o = 0$.

a) Then Eq. (X-122) gives us the value of r :

$$r_o = r_{\min} = a(1 - \varepsilon) .$$

b) From this, the starting Cartesian displacements are

$$x_o = r_o \cos \theta_o = r_o = r_{\min} \quad (\text{X-168})$$

$$y_o = r_o \sin \theta_o = 0 . \quad (\text{X-169})$$

c) The velocity of an object in an elliptical orbit is given by Eq. (X-154). For $r = r_{\min}$, we have

$$v_o = \sqrt{G(M_{\odot} + m) \left(\frac{2}{r_{\min}} - \frac{1}{a} \right)} . \quad (\text{X-170})$$

As such, the initial velocities in Cartesian coordinates are given by

$$v_{x,o} = 0 \quad (\text{X-171})$$

$$v_{y,o} = v_o \quad (\text{X-172})$$

since at $\theta = 0$ (*i.e.*, $r = r_{\min}$), the orbital velocity vector points entirely in the $+y$ direction (see Figure X-10).

Note that this will cause the mass to move in the counter-clockwise direction as drawn in Figure X-10.

7. One then numerically solves Eqs. (X-164) through (X-167) using the Euler-Cromer or 4th order Runge-Kutta (RK4) methods. The best results are usually obtained with RK4 with an adaptive grid (*i.e.*, time interval). See the *Numerical Recipes* books for details.