Sorting index and Mahonian-Stirling pairs for labeled forests

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Permutations

Definition

For a permutation $\sigma = \sigma_1\sigma_2 \ldots \sigma_n \in S_n$, a pair (σ_i, σ_j) is called an inversion if $i < j$ and $\sigma_i > \sigma_j$. The number of inversions of σ is denoted by $\text{inv}(\sigma)$.

Definition

For a permutation $\sigma = \sigma_1\sigma_2 \ldots \sigma_n \in S_n$, Rl-min $= \{\sigma_i | \sigma_i < \sigma_j$ for all $i < j\}$, and $\text{rl-min}(\sigma) = |\{\text{Rl-min}(\sigma)\}|$.

Example

$\sigma = 15324 \qquad \text{inv}(\sigma) = 4 \qquad \text{Rl-min}(\sigma) = \{1, 2, 4\}$

Permutations

Definition (Petersen 2011)

For a permutation $\sigma \in S_n$ there is a unique decomposition $\sigma = (i_1, j_i)(i_2, j_2)\dots(i_k, j_k)$ such that $j_1 < j_2 < \cdots < j_k$ and $i_1 < j_1, i_2 < j_2, \dots, i_k < j_k$. The sorting index is defined by $\operatorname{sor}(P, w) = \sum_{r=1}^{n}(j_r - i_r)$.

Definition

The set of minimal elements of the cycles of a permutation σ is denoted by $\operatorname{Cyc}(\sigma)$ and $\operatorname{cyc}(\sigma) = |\operatorname{Cyc}(\sigma)|$.

Example

$\sigma = 15324 = (1)(254)(3)$ $\quad \operatorname{sor}(\sigma) = 5$ $\quad \operatorname{Cyc}(\sigma) = \{1, 2, 3\}$

Equidistribution Property

Theorem

$$\sum_{\sigma \in S_n} q^{\mathrm{inv}(\sigma)} \prod_{i \in \mathrm{Rl\text{-}min}(\sigma)} t_i = t_1(q + t_2)\cdots(q^{n-1} + \cdots q + t_n)$$

$$\sum_{\sigma \in S_n} q^{\mathrm{sor}(\sigma)} \prod_{i \in \mathrm{Cyc}(\sigma)} t_i = t_1(q + t_2)\cdots(q^{n-1} + \cdots q + t_n)$$

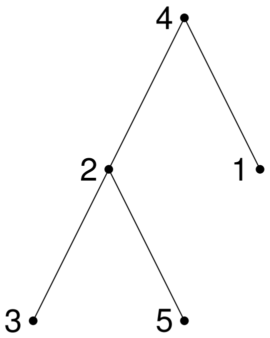
- Björner and Wachs 1991
- Petersen 2011
- Poznanovik 2014

Forest Notation

- A labeling of a forest P with size n is a bijection $w : P \to \{1, 2, \ldots, n\}$.
- The notation $\mathcal{W}(P)$ is used to denote the set of all possible labelings of the forest P.
- The notation $[n]$ will denote the polynomial $q^{n-1} + \cdots + q + 1$.

Forest Notation

- The hook length at x, denoted h_x is the size of the subtree rooted at x.
- Order the vertices of P in increasing hook length and break ties from left to right.



Inversions and Bottom-to-Top Maxima

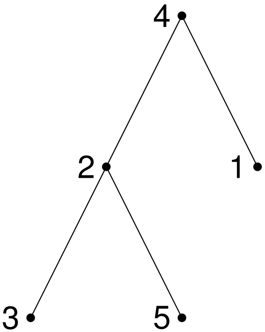
Definition

$\text{inv}(P, w) = |\{(x, y) | x <_P y \text{ and } w(x) > w(y)\}|$ where $w(x)$ is the label of the vertex x.

Definition

$\text{Bt-max}(P, w) = \{i | w(x_i) > w(x_j) \text{ for all } x_j <_P x_i\}$ and $\text{bt-max}(P, w) = |\text{Bt-max}(P, w)|$.

Example



$\mathrm{inv}\,(P, w) = 3 \qquad \text{Bt-max}\,(P, w) = \{1, 2, 3\}$

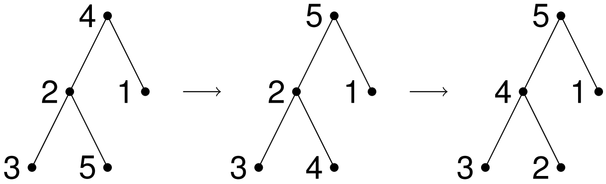
Sorting Index

Definition

The algorithm for computing the sorting index is as follows

```
sor=0
for i in range(n,1,-1)
begin
   In the tree that contains the vertex y
   with label i find the largest vertex, x,
   such that x >_P y and w(x) < i and
   interchange the labels of those two
   vertices sor = sor + w_y(y) - w_y(x)
   call this new labeling w
end
```

Example

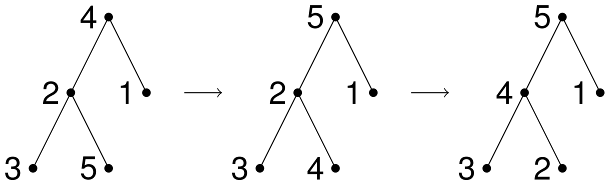


$$\operatorname{sor}(P, w) = 1 + 2 = 3$$

Cycles

Definition

For a forest P with labeling w consider the sorted labeling w'. Then $\mathrm{Cyc}(P, w) =$
$\{i | w'(x_i)$ is the minimum element in a cycle of $w \circ (w')^{-1}\}$ and $\mathrm{cyc}(P, w) = |\mathrm{Cyc}(P, w)|$.



$$w \circ w'^{-1} = (1)(254)(3) \qquad \mathrm{Cyc}(P, w) = \{1, 2, 3\}$$

Equidistribution Property

Theorem (G, Poznanović)

$$\sum_{w \in \mathcal{W}(P)} q^{\operatorname{inv}(P,w)} \prod_{i \in \operatorname{Bt-max}(P,w)} t_i = \frac{n!}{\prod_{i=1}^{n} h_{x_i}} \prod_{i=1}^{n} ([h_{x_i}] - 1 + t_i)$$

$$\sum_{w \in \mathcal{W}(P)} q^{\operatorname{sor}(P,w)} \prod_{i \in \operatorname{Cyc}(P,w)} t_i = \frac{n!}{\prod_{i=1}^{n} h_{x_i}} \prod_{i=1}^{n} ([h_{x_i}] - 1 + t_i)$$

Inversions and Bottom-to-Top Maxima

Theorem (G, Poznanović)

Let P be a forest of size n and let $\mathcal{W}(P)$ be the set of all labelings of P. Then

$$\sum_{w \in \mathcal{W}(P)} q^{\mathrm{inv}\,(P,w)} t^{\text{bt-max}\,(P,w)} = \frac{n!}{\prod_{x \in P} h_x} \prod_{x \in P} ([h_x] - 1 + t).$$

Outline for Proof by Induction

Let P be a forest with trees T_1, T_2, ..., T_k for $k > 1$, and let m_i be the size of T_i. Then

$$\sum_{w \in \mathcal{W}(P)} q^{\text{inv}(P,w)} t^{\text{bt-max}(P,w)}$$

is equal to

$$\binom{n}{m_1, m_2, \ldots, m_k} \prod_{i=1}^{k} \sum_{w \in \mathcal{W}(T_i)} q^{\text{inv}(T_i,w)} t^{\text{bt-max}(T_i,w)}.$$

Outline for Proof by Induction

Now applying the induction hypothesis yields

$$\sum_{w \in \mathcal{W}(P)} q^{\text{inv}(P,w)} t^{\text{bt-max}(P,w)}$$

is equal to

$$\frac{n!}{m_1! m_2! \ldots m_k!} \prod_{i=1}^{k} \frac{m_i!}{\prod_{x \in T_i} h_x} \prod_{x \in T_i} ([h_x] - 1 + t).$$

Outline for Proof by Induction

Now suppose k=1. Let P' be the forest obtained from P by removing the root r. Then every labeling w of P corresponds bijectively with a pair (w', j) where w' is w restricted to P' and $j = w(r)$. Using this we see

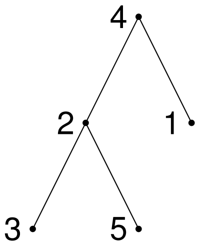
$$\sum_{w \in \mathcal{W}(P)} q^{\mathrm{inv}\,(P,w)}\, t^{\text{bt-max}\,(P,w)}$$

is equal to

$$\sum_{j=1}^{n-1} \sum_{w \in \mathcal{W}(P')} q^{\mathrm{inv}\,(P',w)+n-j}\, t^{\text{bt-max}\,(P',w)} + \sum_{w \in \mathcal{W}(P')} q^{\mathrm{inv}\,(P',w)}\, t^{\text{bt-max}\,(P',w)+1}.$$

A-Code

The A-code for (P, w) is a sequence $(a_1, a_2, \ldots, a_n)$ where $0 \leqslant a_i \leqslant h_{x_i} - 1$ for each $i = 1, 2, \ldots, n$, and $a_i = |\{x_j | w(x_j) > w(x_i) \text{ and } x_j <_P x_i\}|$.



Using the tree P with labeling w from the earlier example the A-code is (0,0,0,2,1).

Inversions and Bottom-to-Top Maxima

Lemma (G, Poznanović)

If the A-code of (P, w) *is* $(a_1, a_2, \ldots, a_n)$*, then* $\mathrm{inv}\,(P, w) = \sum_{i=1}^{n} a_i$*, and* $\text{Bt-max}\,(P, w) = \{i | a_i = 0\}$*.*

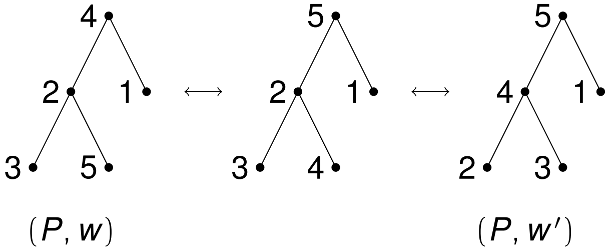
Theorem (G, Poznanović)

Let P *be a forest, then there is a bijection between* $\mathcal{W}(P)$ *and pairs* $(w', (a_1, a_2, \ldots, a_n))$ *where* w' *is a decreasing labeling in* $\mathcal{W}(P)$ *and* $0 \leqslant a_i \leqslant h_{v_i} - 1$ *for all* $i = 1, 2, \ldots, n$*.*

The Bijection

- Given a labeling $w \in \mathcal{W}(P)$ let $(a_1, a_2, \ldots, a_n)$ be the A-code for (P, w).
- Start with $i = n$ and consider the vertex x_i.
- Let the set $A = \{w(x) | x \leqslant_P x_i\}$.
- Find the largest element in A, call this label w_i, and place it on x_i.
- Now relabel the rest of the vertices below x_i with the labels in $A \backslash \{w_i\}$ preserving the order from the original labeling and call the new labeling w.
- Repeat this process with $i = n - 1, n - 2, \ldots, 1$.
- Call the final labeling w' to get the ordered pair (w', A-code(P,w)).

Example



With the A- code $(0, 0, 0, 2, 1)$.

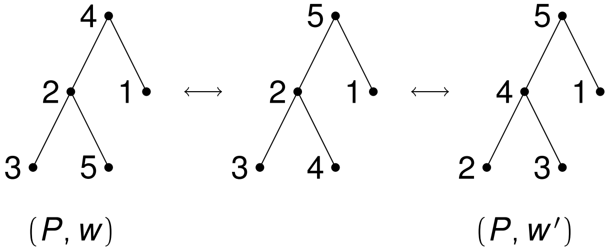
The Inverse

Given the pair $(w', (a_1, a_2, \ldots, a_n))$ we can reconstruct the corresponding labeling with that A-code, w, from w' using the following steps.

- Start with $i = 1$.
- Consider the vertex x_i and let $A = \{w'(x) | x \leqslant_P x_i\}$. Find the $a_i + 1^{st}$ largest label in A and call it w_i'.
- Relabel the vertex x_i with w_i' and the rest of the vertices below x_i with the labels in $A \backslash \{w_i'\}$ preserving the order from the original labeling.
- Call this new labeling w'.
- Repeat this process for $i = 2, 3, \ldots, n$.

The final labeling will be the labeling w with the given A-code.

Example



With the A- code $(0, 0, 0, 2, 1)$.

Inversions and Bottom-to-Top Maxima

This bijection gives the generating function that encodes the positions of the bottom-to-top maxima.

Corollary (G, Poznanović)

Let P be a forest of size k, then

$$\sum_{w \in \mathcal{W}(P)} q^{\operatorname{inv}(P,w)} \prod_{i \in \text{Bt-max}(P,w)} t_i = \frac{n!}{\prod_{i=1}^{n} h_{x_i}} \prod_{i=1}^{n} ([h_{x_i}] - 1 + t_i).$$

Inversions and Bottom-to-Top Maxima

Theorem (G, Poznanović)

Let P be a tree of size n with labeling w and only one leaf. Let $\sigma = w(x_1)w(x_2)\cdots w(x_n)$. Then $\text{Bt-max}(P, w) = \text{Rl-min}(\sigma^{-1})$, *and* $\text{inv}(P, w) = \text{inv}(\sigma^{-1})$.

Therefore the previous Corollary generalizes the following result for permutations.

$$\sum_{\sigma \in S_n} q^{\text{inv}(\sigma)} \prod_{i \in \text{Rl-min}(\sigma)} t_i = \prod_{i=1}^{n} ([i] - 1 + t_i)$$

Sorting Index and Cycles

Theorem (G, Poznanović)

Let P be a forest of size n and let $\mathcal{W}(P)$ be the set of all labelings of P. Then

$$\sum_{w \in \mathcal{W}(P)} q^{\operatorname{sor}(P,w)} t^{\operatorname{cyc}(P,w)} = \frac{n!}{\prod_{x \in P} h_x} \prod_{x \in P} ([h_x] - 1 + t).$$

Outline for Proof by Induction

Let P be a forest with trees T_1, T_2, ..., T_k for $k > 1$, and let m_i be the size of T_i. Then

$$\sum_{w \in \mathcal{W}(P)} q^{\text{sor}\,(P,w)}\, t^{\text{cyc}\,(P,w)}$$

is equal to

$$\binom{n}{m_1, m_2, \ldots, m_k} \prod_{i=1}^{k} \sum_{w \in \mathcal{W}(T_i)} q^{\text{sor}\,(P,w)}\, t^{\text{cyc}\,(P,w)}.$$

Outline for Proof by Induction

Now applying the induction hypothesis yields

$$\sum_{w \in \mathcal{W}(P)} q^{\mathrm{sor}(P,w)} t^{\mathrm{cyc}(P,w)}$$

is equal to

$$\frac{n!}{m_1!m_2!\dots m_k!} \prod_{i=1}^{k} \frac{m_i!}{\prod_{x \in T_i} h_x} \prod_{x \in T_i} ([h_x] - 1 + t).$$

Outline for Proof by Induction

Now suppose that k=1. Let P' be the forest obtained from removing the root, r, from the forest P=T_1. Now every labeling, w, of P corresponds bijectively to a pair (j, w') where $j = w(r)$, and w' is found be restricting w to P' and replacing the weight n with j. Using this bijection we see

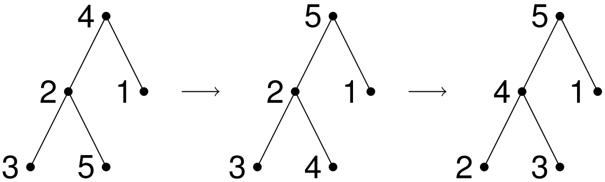
$$\sum_{w \in \mathcal{W}(P)} q^{\text{sor}(P,w)} t^{\text{cyc}(P,w)}$$

is equivalent to

$$\sum_{j=1}^{n-1} \sum_{w \in \mathcal{W}(P')} q^{\text{sor}(P',w)+n-j} t^{\text{cyc}(P',w)} + \sum_{w \in \mathcal{W}(P')} q^{\text{sor}(P',w)} t^{\text{cyc}(P',w)+1}.$$

B-code

The B-code for (P, w) is a sequence $(b_1, b_2, \ldots, b_n)$ where $0 \leqslant b_i \leqslant h_{v_i} - 1$ for each $i = 1, 2, \ldots, n$, and b_i is the amount contributed to the sorting index by the vertex v_i.



(P, w) (P, w')

B-code = (0,0,0,2,1).

Sorting Index and Cycles

Lemma (G, Poznanović)

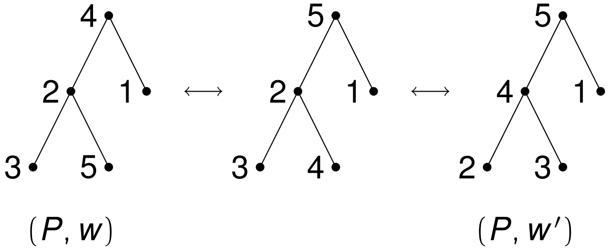
If the B-code of (P, w) *is* $(b_1, b_2, \ldots, b_n)$ *then* $\sum_{i=1}^{n} b_i = \operatorname{sor}(P, w)$ *and* $\operatorname{Cyc}(P, w) = \{i | b_i = 0\}$.

Theorem (G, Poznanović)

For a given forest P with size n, there is a bijection between labelings $w \in \mathcal{W}(P)$ *and pairs* $(w', (b_1, b_2, \ldots, b_n))$ *where* w' *is a decreasing labeling in* $\mathcal{W}(P)$ *and* $0 \leqslant b_i \leqslant h_{v_i} - 1$ *for all* $i = 1, 2, \ldots, n$.

The Bijection

Given a labeling $w \in \mathcal{W}(P)$ to get the corresponding ordered pair just apply the sorting algorithm and the resulting labeling is w'. Then find the B-code to get the pair (w',B-code(P, w)).



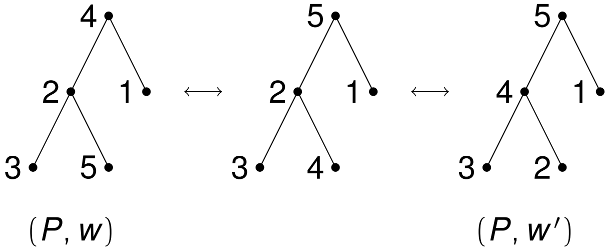
With the B-code (0,0,0,2,1).

The Inverse

Now given a pair $(w', (b_1, b_2, \ldots, b_n))$ we can recover the original labeling w with the B-code $(b_1, b_2, \ldots, b_n)$ in the following way.

- Start with $i = 1$.
- Consider the vertex x_i and let $B = \{w'(x_i) | x_j \leqslant_P x_i\}$.
- Let $w'(x_j)$ be the $b_i + 1^{st}$ largest element of B.
- Interchange the labels on x_i and x_j and call this new labeling w'.
- Repeat this process for $i = 2, 3, \ldots, n$. The final labeling is w.

Example



With B-code $(0, 0, 0, 2, 1)$.

Sorting Index and Cycles

This bijection gives a generating function that encodes the elements of Cyc.

Corollary (G, Poznanović)

Let P be a forest of size n, then

$$\sum_{w \in \mathcal{W}(P)} q^{\mathrm{sor}\,(P,w)} \prod_{i \in \mathrm{Cyc}\,(P,w)} t_i = \frac{n!}{\prod_{i=1}^{n} h_{x_i}} \prod_{i=1}^{n} ([h_{x_i}] - 1 + t_i).$$

Sorting Index and Cycles

Theorem (G, Poznanović)

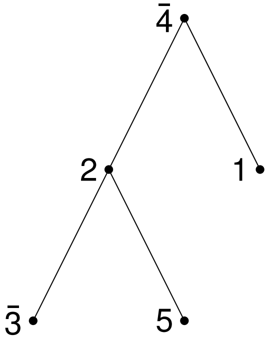
Let P be a tree of size n with only one leaf, and a labeling w. Let $\sigma = w(x_1)w(x_2)\cdots w(x_n)$, then $\operatorname{sor}(P, w) = \operatorname{sor}(\sigma^{-1})$ *and* $\operatorname{Cyc}(P, w) = \operatorname{Cyc}(\sigma^{-1})$.

Therefore the previous Corollary generalizes the result for permutations.

$$\sum_{\sigma \in S_n} q^{\operatorname{sor}(\sigma)} \prod_{i \in \operatorname{Cyc}(\sigma)} t_i = \prod_{i=1}^{n} ([i] - 1 + t_i)$$

Signed Labeled Forests

A signed labeling is a map $w : P \to \{-n, -n+1, \ldots, n\} \setminus \{0\}$ such that if $i \in w(P)$ then $-i \notin w(P)$. Note that $-i$ is denoted with $\bar{i}$.



Signed Labeled Forests

Theorem (G, Poznanović)

$$\sum_{w \in \mathcal{W}_B(P)} q^{\mathrm{inv_B}(P,w)} \prod_{i \in \mathrm{Bt\text{-}max_B}(P,w)} t_i = \frac{n!}{\prod_{i=1}^n h_{x_i}} \prod_{i=1}^n ([2h_{x_i}] - 1 + t_i)$$

$$\sum_{w \in \mathcal{W}_B(P)} q^{\mathrm{sor_B}(P,w)} \prod_{i \in \mathrm{Cyc_B}(P,w)} t_i = \frac{n!}{\prod_{i=1}^n h_{x_i}} \prod_{i=1}^n ([2h_{x_i}] - 1 + t_i)$$

Major Index and Cyclic Bottom-to-Top Maxima

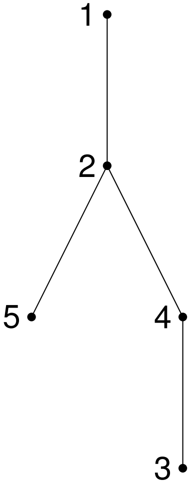
Definition

For a labeled forest P, with labeling w
$\mathrm{Des}\,(P, w) = \{u \in P | w(u) > w(v), v \text{ is the parent of } u\}$
$\mathrm{maj}\,(P, w) = \sum_{u \in \mathrm{Des}\,(P,w)} h_u$.

Definition

In a forest P of size n and labeling w a vertex is a cyclic bottom-to-top maximum if its label is a bottom-to-top maximum with respect to the cyclic ordering of the integers from 1 to n beginning with the label of its parent. Let Cbt-max denote the set of such vertices, and cbt-max $= |$Cbt-max$|$.

Example



$\text{maj}\,(P, w) = 7 \qquad \text{Cbt-max}\,(P, w) = \{1, 2, 3\}$

Major Index and Cyclic Bottom-to-Top Maxima

Theorem (G, Poznanović)

Let P be a forest of size n, then

$$\sum_{w \in \mathcal{W}(P)} q^{\mathrm{maj}\,(P,w)} \prod_{i \in \text{Cbt-max}\,(P,w)} t_i = \frac{n!}{\prod_{i=1}^{n} h_{x_i}} \prod_{i=1}^{n} ([h_{x_i}] - 1 + t_i).$$

Future Work

- Looking at the distributions of these statistics over even signed trees.
- Defining a graphical sorting index.
- Finding pairs and generating functions for the graphical major index, inversions, and sorting index.

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