Pattern Avoidance on $k$-ary Heaps
(Work in Progress - aren’t they all)

Derek Levin, Lara Pudwell, Manda Riehl, and Andrew Sandberg

University of Wisconsin - Eau Claire, Valparaiso University

Permutation Patterns - July 10, 2014
Once upon a time...
Motivation

Sophia Yakoubov: Paris 2013
Motivation

Sophia Yakoubov: Paris 2013  Pattern Avoidance on Combs
Motivation

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Pattern Avoidance on $k$-ary Heaps
Something like combs, but not combs

Heaps!
Something like combs, but not combs

Heaps!
A *complete binary tree* is a tree where each node has 2 or fewer children, all levels except possibly the last are completely full, and the last level has all its nodes to the left side.
A heap is a complete binary tree labelled with \( \{1, \ldots, n\} \) such that every child has a larger label than its parent.
A heap is a complete binary tree labelled with \{1, \ldots, n\} such that every child has a larger label than its parent.
Where's the pattern?

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Pattern Avoidance on $k$-ary Heaps
Where’s the pattern?

1 4 2 6 8 3 5 7 9 10

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Pattern Avoidance on k-ary Heaps
Where’s the pattern?

Notice this heap contains 123, 132, 213, 231, 312.
Where’s the pattern?

Notice this heap contains 123, 132, 213, 231, 312. But it avoids 321.
$k$-ary Heaps

Pattern Avoidance on $k$-ary Heaps

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Pattern Avoidance on $k$-ary Heaps
$k$-ary Heaps

This heap avoids 231.
Forests of Heaps

Pattern Avoidance on $k$-ary Heaps
Forests of Heaps

5 10 9 2 11 6 1 3 7 4 12 8

Notice each tree avoids 213, 231, 312, 321, but the forest contains 213, 231, 312, 321.
Crunch the numbers, cross your fingers

<table>
<thead>
<tr>
<th>Heaps Avoiding:</th>
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<tbody>
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Heaps Avoiding \((231, 312)\)

Narayana-Zidek-Capell Numbers:
1, 1, 2, 3, 6, 11, 22, 42, 84, 165, 330, …

Given by relation:
\[ a_n = 2a_{n-1} \text{ if } n \text{ even} \]
\[ a_n = 2a_{n-1} - a_{\frac{n-1}{2}} \text{ if } n \text{ odd} \]
Narayana-Zidek-Capell Numbers:
1, 1, 2, 3, 6, 11, 22, 42, 84, 165, 330, …

Insertion argument:
Narayana-Zidek-Capell Numbers:
1, 1, 2, 3, 6, 11, 22, 42, 84, 165, 330, \ldots
Insertion argument:
Insert $n + 1$, but leave it an increasing tree
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Heaps Avoiding \((231, 312)\)

Narayana-Zidek-Capell Numbers:
1, 1, 2, 3, 6, 11, 22, 42, 84, 165, 330, \ldots

Insertion argument:
Insert \(n + 1\), but leave it an increasing tree

\[\begin{array}{c}
3 & 5 \\
2 & 4 \\
1 \\
\end{array}\] \[\begin{array}{c}
4 & 3 & 5 \\
6 & 1 \\
2 \\
\end{array}\]
Lemma

The vertex labelled $n$ is always a leaf. After insertion, the vertex labelled $n + 1$ is always a leaf.
Lemma
In order to avoid 231 and 312, $n + 1$ must be inserted directly before $n$ or at the end.
Heaps Avoiding $(231, 312)$: $n + 1$ must be right before $n$, or at end

Proof of Lemma:
- $n + 1$ at last leaf: OK
- $n + 1$ right before $n$: OK
Heaps Avoiding \((231, 312)\): \(n + 1\) must be right before \(n\), or at end

Proof of Lemma:

- If \(n + 1\) is inserted more before \(n\), we create a 312.
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Heaps Avoiding $(231, 312)$: $n + 1$ must be right before $n$, or at end

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Heaps Avoiding \((231, 312)\): \(n + 1\) must be right before \(n\), or at end

Proof of Lemma:

- If \(n + 1\) is inserted after \(n\), but not at the end, we create a 231.
Heaps Avoiding \((231, 312)\): \(n + 1\) must be right before \(n\), or at end

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Heaps Avoiding $(231, 312)$: $n + 1$ must be right before $n$, or at end

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Heaps Avoiding $(231, 312)$: $n + 1$ must be right before $n$, or at end

Proof of Lemma:
- If $n + 1$ is inserted after $n$, but not at the end, we create a 231.
Easy Case: $n$ is even. So the new leaf is the sibling of a current leaf. Internal nodes stay internal, leaves stay leaves.
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We can put \( n + 1 \) at the (new) last leaf, or we can insert it right before \( n \) and push everything along.
Easy Case: \( n \) is even. So the new leaf is the sibling of a current leaf. Internal nodes stay internal, leaves stay leaves.

We can put \( n + 1 \) at the (new) last leaf, or we can insert it right before \( n \) and push everything along.

\[ a_n = 2a_{n-1}. \]
Second Case: $n$ is odd. So the new leaf is child of a former leaf. Is it possible that we push a small label to be a child over a larger label that was a leaf?
Inserting $n + 1$ at the last leaf:
Still OK!
Heaps Avoiding (231, 312)

Inserting \( n + 1 \) anywhere except the first or last leaf:
Heaps Avoiding (231, 312)

Inserting $n + 1$ anywhere except the first or last leaf:
Inserting $n + 1$ anywhere except the first or last leaf:

We must already have had a $231$, namely $b\ n\ a$. 

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Pattern Avoidance on $k$-ary Heaps
Inserting $n+1$ at the first leaf:
Inserting $n + 1$ at the first leaf:
Heaps Avoiding (231, 312)

Inserting $n + 1$ at the first leaf:

We don’t want to count this, even though it may still avoid 231, 312.
How many shouldn’t we count?
Since $n$ was on the first leaf, all other leaf labels are in decreasing order.
How many shouldn’t we count?
Since $n$ was on the first leaf, all other leaf labels are in decreasing order.
The subtree obtained by removing all leaves needs to avoid 231,312.
How many shouldn’t we count?

Since \( n \) was on the first leaf, all other leaf labels are in decreasing order.

The subtree obtained by removing all leaves needs to avoid 231,312.

There are \( \frac{n-1}{2} \) nodes on that subtree.
How many shouldn’t we count?
Since $n$ was on the first leaf, all other leaf labels are in decreasing order.
The subtree obtained by removing all leaves needs to avoid 231,312.
There are $\frac{n-1}{2}$ nodes on that subtree.
Thus $a_n = 2a_{n-1} - a_{n-1}^{\frac{n-1}{2}}$ when $n$ is odd.
How many shouldn’t we count?
Since $n$ was on the first leaf, all other leaf labels are in decreasing order.
The subtree obtained by removing all leaves needs to avoid 231,312.
There are $\frac{n-1}{2}$ nodes on that subtree.
Thus $a_n = 2a_{n-1} - a_{\frac{n-1}{2}}$ when $n$ is odd.
So we have the same recurrence relation as the Narayana-Zidek-Capell numbers.
Test yourself!
Heaps Avoiding 231

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Pattern Avoidance on $k$-ary Heaps
Many excellent designs for a new banner were submitted. We will use the best of them in rotation.

Search: seq:1,1,2,3,7,14,37,80,222

Sorry, but the terms do not match anything in the table.

If your sequence is of general interest, please submit it using the form provided and it will (probably) be added to the OEIS! Include a brief description and if possible enough terms to fill 3 lines on the screen. We need at least 4 terms.

Search completed in 0.100 seconds
Let’s look at where the $n$ appears. (Definitely on a leaf)
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All the labels before $n$ are less than all labels after $n.$
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The labels on the leaves after $n$ can be arranged in Catalan many ways.
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All the labels before $n$ are less than all labels after $n$.

The labels on the leaves after $n$ can be arranged in Catalan many ways.

The subheap before $n$ is a smaller case of a heap avoiding 231.
Let $b_n$ be the number of heaps with $n$ nodes avoiding 231.
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Let $i$ be the number of leaves after $n$. $0 \leq i \leq \left\lfloor \frac{n-1}{2} \right\rfloor$. 
Let $b_n$ be the number of heaps with $n$ nodes avoiding 231.

Let $i$ be the number of leaves after $n$. $0 \leq i \leq \lfloor \frac{n-1}{2} \rfloor$.

$$b_n = \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} C_i b_{n-i-1}$$
$k$-ary forests avoiding 132

\begin{center}
\begin{tikzpicture}
  \foreach \i in {1,...,6}{\node at \i (\i) {};}\node at 7 (7) ;\node at 8 (8) ;\node at 9 (9) ;\node at 10 (10) ;\node at 11 (11) ;\node at 12 (12) ;\node at 5 (5) ;\node at 3 (3) ;\node at 1 (1) ;\node at 2 (2) ;
  \foreach \i in {1,2,3,4,5,6}{\node at \i (\i) {};}\node at 7 (7) ;\node at 8 (8) ;\node at 9 (9) ;\node at 10 (10) ;\node at 11 (11) ;\node at 12 (12) ;\node at 5 (5) ;\node at 3 (3) ;\node at 1 (1) ;\node at 2 (2) ;
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\end{tikzpicture}
\end{center}
Obvious statements: Each tree avoids 132.
Obvious statements: Each tree avoids 132. The roots of each tree avoid 132.
$k$-ary forests avoiding 132
Lemma

Knowing the roots is enough!
The number of forests of \( t \) heaps each with \( v \) vertices that avoid 132 is given by \( \frac{1}{vt+1} (v+1)^t \).
The number of forests of $t$ heaps each with $v$ vertices that avoid 132 is given by $\frac{1}{vt+1} \binom{(v+1)t}{t}$.

Proof method: Bijection to the number of paths under the line $y = vx$ from $(0,0)$ to $(t, vt)$ using steps $(0,1)$ and $(1,0)$. 
Examples of bijection

\[ y = 2x \]

1

2

5

Pattern Avoidance on \( k \)-ary Heaps
Examples of bijection

Make a string of the level step heights in reverse order: 4 1 1 0
Examples of bijection

Make a string of the level step heights in reverse order: 4 1 1 0
Add one to each element of the string. 5 2 2 1
Examples of bijection

Make a string of the level step heights in reverse order: 4 1 1 0
Add one to each element of the string: 5 2 2 1
The smallest number currently unused in the forest that is greater than or equal to the next element of the string gives the next root.
Examples of bijection
Examples of bijection

\[ y = 2x \]

1 2 5

6 3 7 8

5 2 4 1
Examples of bijection

4 5 7 8 2 9
3 6 1
Examples of bijection

Make a string, starting with the root of the first heap.

3
Examples of bijection

Make a string, starting with the root of the first heap. Repeat the same number in the string for each root as long as the permutation is increasing.

3 → 3 3
Examples of bijection

Make a string, starting with the root of the first heap. Repeat the same number in the string for each root as long as the permutation is increasing. If the permutation has a descent, the next root is the next entry in the string.

$$3 \rightarrow 3 \ 3 \rightarrow 3 \ 3 \ 1$$
Examples of bijection

Make a string, starting with the root of the first heap. Repeat the same number in the string for each root as long as the permutation is increasing. If the permutation has a descent, the next root is the next entry in the string. Subtract 1 from each element of the string. These are your sequence of level steps. 3 → 3 3 → 3 3 1 → 2 2 0
Examples of bijection

Derek Levin, Lara Pudwell, Manda Riehl, and Andrew Sandberg

Pattern Avoidance on $k$-ary Heaps
Examples of bijection

图案避免在 $k$-元堆上的应用
Corollary

Let $\sigma$ be a permutation of length $nm$ composed of a concatenation of $m$ increasing sequences of length $n$. The number of such $\sigma$ that avoid 132 is $\frac{1}{nm+1} \binom{(n+1)m}{nm+1}$. 
<table>
<thead>
<tr>
<th>Heaps Avoiding:</th>
<th>Sequence</th>
<th>OEIS#</th>
</tr>
</thead>
<tbody>
<tr>
<td>∅</td>
<td>1, 1, 2, 3, 8, 20, 80, 210, 896...</td>
<td>A056971</td>
</tr>
<tr>
<td>123</td>
<td>1, 1, 1, 0, 0, 0, 0, 0...</td>
<td>A000004</td>
</tr>
<tr>
<td>132</td>
<td>1, 1, 1, 1, 1, 1, 1...</td>
<td>A000012</td>
</tr>
<tr>
<td>213</td>
<td>1, 1, 2, 2, 5, 5, 14, 14, 42...</td>
<td>A208355</td>
</tr>
<tr>
<td>231 = 312</td>
<td>1, 1, 2, 3, 7, 14, 37, 80, 222,...</td>
<td>Soon in OEIS!</td>
</tr>
<tr>
<td>321</td>
<td>1, 1, 2, 3, 7, 16, 45, 111, 318...</td>
<td>OPEN</td>
</tr>
<tr>
<td>{213, 231} =</td>
<td>1, 1, 2, 2, 4, 4, 8, 8, 16...</td>
<td>A016116</td>
</tr>
<tr>
<td>{213, 312}</td>
<td></td>
<td></td>
</tr>
<tr>
<td>{213, 321}</td>
<td>1, 1, 2, 2, 4, 4, 7, 7, 11...</td>
<td>A000124</td>
</tr>
<tr>
<td>{231, 312} =</td>
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<td>A002083</td>
</tr>
<tr>
<td>{231, 321}</td>
<td></td>
<td></td>
</tr>
<tr>
<td>{231, 312, 321}</td>
<td>1, 1, 2, 3, 5, 8, 13...</td>
<td>A000045</td>
</tr>
</tbody>
</table>
What about $k$-ary?

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</tr>
</thead>
<tbody>
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<td>A056971</td>
</tr>
<tr>
<td>123</td>
<td>1, 1, 1, 0, 0, 0, 0, 0...</td>
<td>A000004</td>
</tr>
<tr>
<td>132</td>
<td>1, 1, 1, 1, 1, 1, 1, 1...</td>
<td>A000012</td>
</tr>
<tr>
<td>213</td>
<td>1, 1, 2, 2, 5, 5, 14, 14, 42...</td>
<td>A208355</td>
</tr>
<tr>
<td>231 = 312</td>
<td>1, 1, 2, 3, 7, 14, 37, 80, 222, ...</td>
<td>Soon to be in OEIS!</td>
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<td>{213, 231} =</td>
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What next?

- Trees that aren’t heaps
What next?

- Trees that aren’t heaps
- Unary-binary, binary, $k$-ary
What next?

- Trees that aren’t heaps
- Unary-binary, binary, $k$-ary
- Slightly different question: How many permutations avoid $\sigma$ can be realized as trees?
Thank You!

- Anant Godbole
- Permutation Patterns 2014 Organizers
- UWEC Department of Mathematics
- UWEC Office of Research and Sponsored Programs
