Chapter 1. Limits and Continuity

1.2. Finding Limits and One-Sided Limits

Theorem 1. Limit Rules.

If L, M, c, and k are real numbers and

$$\lim_{x\to c} f(x) = L \quad \text{and} \quad \lim_{x\to c} g(x) = M, \quad \text{then}$$

1. *Sum Rule*: $\lim_{x\to c}(f(x) + g(x)) = L + M$.

2. *Difference Rule*: $\lim_{x\to c}(f(x) - g(x)) = L - M$.

3. *Product Rule*: $\lim_{x\to c}(f(x) \cdot g(x)) = L \cdot M$.

4. *Constant Multiple Rule*: $\lim_{x\to c}(k \cdot f(x)) = k \cdot L$.

5. *Quotient Rule*: $\lim_{x\to c} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}, \quad M \neq 0$.

6. *Power Rule*: If r and s are integers, $s \neq 0$, then

$$\lim_{x\to c}(f(x))^{r/s} = L^{r/s}$$

provided that $L^{r/s}$ is a real number **AND** $L > 0$.

Note. We must have $L > 0$ in part 6 of Theorem 1 since $\lim_{x \to 0} \sqrt{x}$ does not exist. **NOTICE THAT THERE IS AN ERROR IN THE TEXT!!!**

Proof of Theorem 1, part 1. We wish to prove $\lim_{x \to c}(f(x) + g(x)) = L + M$ under the assumptions $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = M$. Let $\epsilon > 0$ be given. Then $\epsilon/2 > 0$ and there exists $\delta_1 > 0$ such that for all x with $0 < |x - c| < \delta_1$ we have $|f(x) - L| < \epsilon/2$. Similarly, there exists $\delta_2 > 0$ such that for all x with $0 < |x - c| < \delta_2$ we have $|g(x) - M| < \epsilon/2$. Therefore we choose $\delta = \min\{\delta_1, \delta_2\}$. Then for $0 < |x - c| < \delta$ we have

$$\begin{aligned}
|(f(x) + g(x)) - (L + M)| &\leq |(f(x) - L) + (g(x) - M)| \\
&\leq |f(x) - L| + |g(x) - M| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon.
\end{aligned}$$

This proves the result. *Q.E.D.*

Note. For proofs of parts 2 through 5 of Theorem 1, see pages 1147–1148. Notice that the text does not provide a proof of part 6 (since as the text states it, it is false!).

Example. Page 109 number 8.

Theorem 2. Limits of Polynomials Can Be Found by Substitution.

If $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ then

$$\lim_{x \to c} P(x) = P(c) = a_n c^n + a_{n-1}c^{n-1} + \cdots + a_0.$$

Theorem 3. Limits of Rational Functions Can Be Found by Substituting IF the Limit of the Denominator Is Not Zero.

If $P(x)$ and $Q(x)$ are polynomials and $Q(c) \neq 0$, then

$$\lim_{x \to c} \frac{P(x)}{Q(x)} = \frac{P(c)}{Q(c)}.$$

Example. Page 109 number 12a.

Theorem. Dr. Bob's Theorem. (NOT IN 10TH EDITION!)

If $f(x) = g(x)$ for all x in an open interval containing c, except possibly c itself, then

$$\lim_{x \to c} f(x) = \lim_{x \to c} g(x)$$

provided these limits exist.

Note. We have to be careful in our dealings with functions! Notice that $f(x) = \dfrac{x(x-1)}{x-1}$ and $g(x) = x$ are **NOT** the same functions! They do not even have the same domains. Therefore we cannot in general say $\dfrac{x(x-1)}{x-1} = x$. However, this equality holds if x lies in the domains of the functions. We *can* say:

$$\frac{x(x-1)}{x-1} = x \textbf{ IF } x \neq 1.$$

We can also say $f(x) = g(x)$ **IF** $x \neq 1$. If we are concerned with limits as x approaches 1, then from the definition, x **IS NOT EQUAL TO 1** (but near 1). Therefore we can say $\lim_{x\to 1} f(x) = \lim_{x \to 1} g(x)$. We have not said that the functions are equal, but that their limits are.

Example. Page 109 number 14b.

Example. Page 109 number 14a.

Theorem 4. Sandwich Theorem.

Suppose that $g(x) \leq f(x) \leq h(x)$ for all x in some open interval containing c, except possibly at $x = c$ itself. Suppose also that

$$\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L.$$

Then $\lim_{x \to c} f(x) = L$.

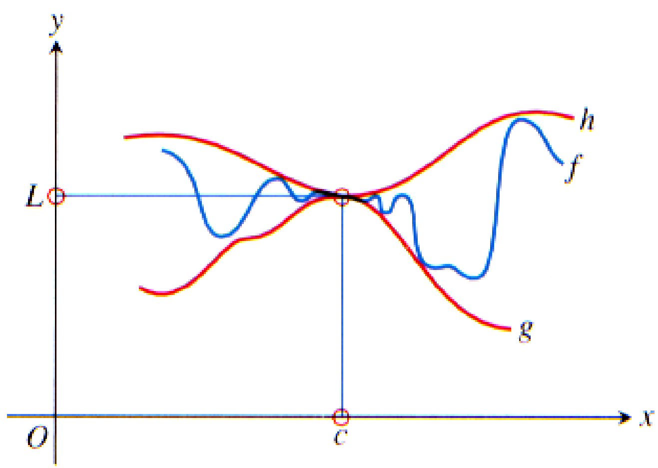

Figure 1.2.17, page 102

Example. Page 109 number 16.

Definition. Informal Definition of Right-Hand and Left-Hand Limits.

Let $f(x)$ be defined on an interval (a, b), where $a < b$. If $f(x)$ approaches arbitrarily close to L as x approaches a from within that interval, then we say that f has *right-hand limit* L at a, and write

$$\lim_{x \to a^+} f(x) = L.$$

Let $f(x)$ be defined on an interval (c, a), where $c < a$. If $f(x)$ approaches arbitrarily close to M as x approaches a from within the interval (c, a), then we say that f has *left-hand limit* M at a, and we write

$$\lim_{x \to a^-} f(x) = M.$$

Definition. Formal Definitions of One Sided Limits. (NOT IN 10TH EDITION!)

We say that $f(x)$ has *right-hand limit* L *at* x_0, and write

$$\lim_{x \to x_0^+} f(x) = L$$

if for every number $\epsilon > 0$ there exists a corresponding number $\delta > 0$ such that for all x

$$x_0 < x < x_0 + \delta \quad \Rightarrow \quad |f(x) - L| < \epsilon.$$

We say that $f(x)$ has *left-hand limit* L *at* x_0, and write

$$\lim_{x \to x_0^-} f(x) = L$$

if for every number $\epsilon > 0$ there exists a corresponding number $\delta > 0$ such that for all x

$$x_0 - \delta < x < x_0 \quad \Rightarrow \quad |f(x) - L| < \epsilon.$$

Example. Consider limits as x approaches -1 and $+1$ for $f(x) = \sqrt{1 - x^2}$.

Theorem 5. Relation Between One-Sided and Two-Sided Limits

A function $f(x)$ has a limit as x approaches c if and only if it has left-hand and right-hand limits there and these one-sided limits are equal:

$$\lim_{x \to c} f(x) = L \quad \Leftrightarrow \quad \lim_{x \to c^-} f(x) = L \quad \text{and} \quad \lim_{x \to c^+} f(x) = L.$$

Example. Page 110 number 26.

Example. Page 110 number 42.

Theorem 6.

For θ in radians,

$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1.$$

Proof. Suppose first that θ is positive and less than $\pi/2$. Consider the picture:

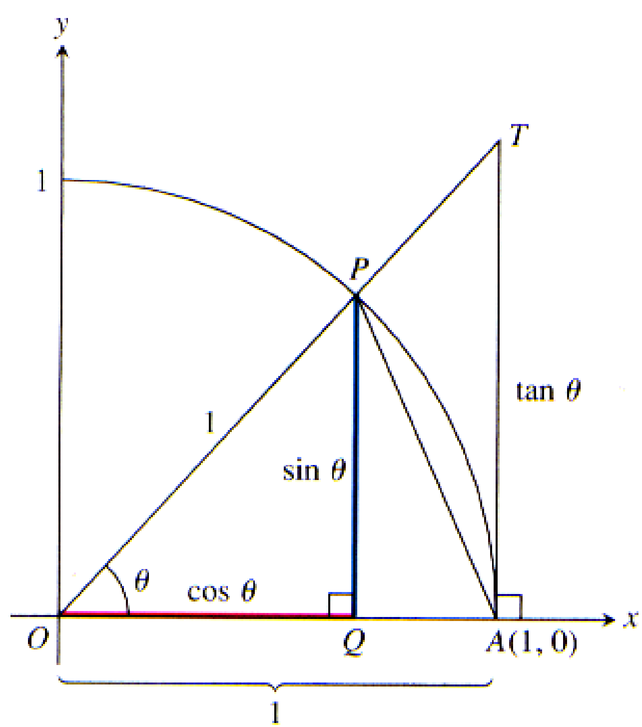

Figure 1.2.25, page 106

Notice that

$$\text{Area } \Delta OAP < \text{area sector } OAP < \text{area } \Delta OAT.$$

We can express these areas in terms of θ as follows:

$$\text{Area } \Delta OAP = \tfrac{1}{2} \text{ base} \times \text{ height} = \tfrac{1}{2}(1)(\sin\theta) = \tfrac{1}{2}\sin\theta$$

$$\text{Area sector } OAP = \tfrac{1}{2}r^2\theta = \tfrac{1}{2}(1)^2\theta = \tfrac{\theta}{2}$$

$$\text{Area } \Delta OAT = \tfrac{1}{2} \text{ base} \times \text{ height} = \tfrac{1}{2}(1)(\tan\theta) = \tfrac{1}{2}\tan\theta.$$

Thus,

$$\frac{1}{2}\sin\theta < \frac{1}{2}\theta < \frac{1}{2}\tan\theta.$$

This last inequality goes the same way if we divide all three terms by the positive number $(1/2)\sin\theta$:

$$1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}.$$

Taking reciprocals reverses the inequalities:

$$\cos\theta < \frac{\sin\theta}{\theta} < 1.$$

Since $\lim_{\theta\to 0^+} \cos\theta = 1$, the Sandwich Theorem gives

$$\lim_{\theta\to 0^+} \frac{\sin\theta}{\theta} = 1.$$

Since $\sin\theta$ and θ are both odd functions, $f(\theta) = \dfrac{\sin\theta}{\theta}$ is an even function and hence $\dfrac{\sin(-\theta)}{-\theta} = \dfrac{\sin\theta}{\theta}$. Therefore

$$\lim_{\theta\to 0^-} \frac{\sin\theta}{\theta} = 1 = \lim_{\theta\to 0^+} \frac{\sin\theta}{\theta},$$

so $\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$ by Theorem 4. *QED*

Example. Page 107 example 10a: Show that $\lim_{h\to 0} \frac{\cos h - 1}{h} = 0$.

Solution. We use the trig identity $\cos h = 1 - 2\sin^2(h/2)$ (a half-angle identity). First, we have

$$\lim_{h\to 0} \frac{\cos h - 1}{h} = \lim_{h\to 0} -\frac{2\sin^2(h/2)}{h}.$$

Now, replacing $h/2$ with θ we get

$$\lim_{h\to 0} \frac{\cos h - 1}{h} = \lim_{h\to 0} -\frac{2\sin^2(h/2)}{h} = -\lim_{\theta\to 0} \frac{\sin\theta}{\theta}\sin\theta = -(1)(0) = 0.$$

QED