# Chapter 4. Integration

## 4.7. Numerical Integration

**Note.** If we start with a regular partition, then we can approximate definite integrals using trapezoids instead of rectangles.

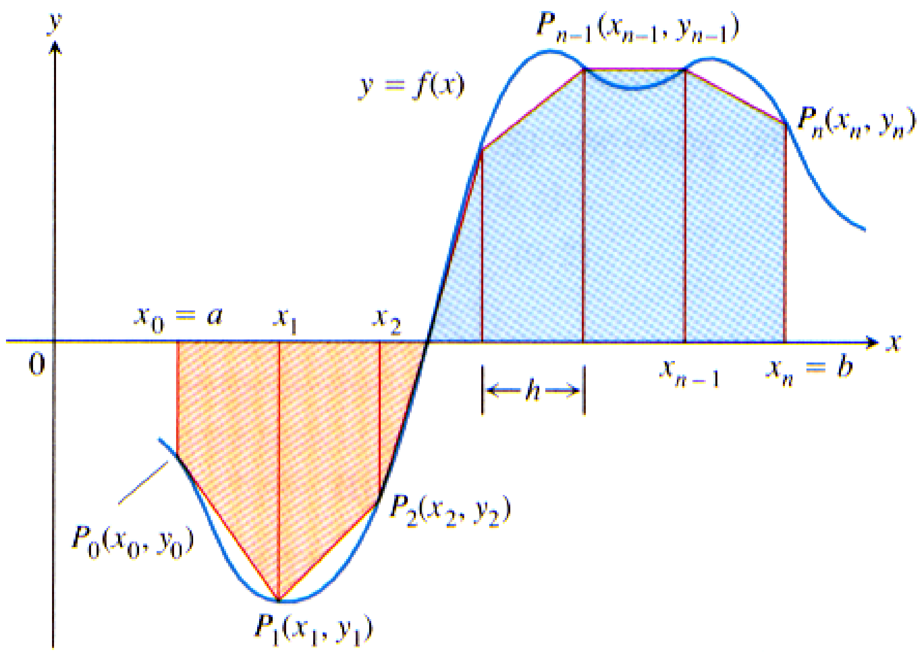


Figure 4.7.24, page 374

We let $h = \Delta x_k = \dfrac{b-a}{n}$ and the area of the $k$th trapezoid is

$$(\text{base}) \times (\text{average height}) = \frac{y_{k-1}+y_k}{2}h = \frac{1}{2}(y_{k-1}+y_k)h.$$

So our estimate is

$$T = \sum_{k=1}^{n} \frac{1}{2}(y_{k-1}+y_k)h = \frac{h}{2}(y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n).$$

**Definition.** In the *Trapezoid Rule*, the integral $\int_a^b f(x)\,dx$, is approximated by

$$T = \frac{h}{2}\left(y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n\right).$$

This approximation is based on a regular partition of $[a, b]$ where $\Delta x_k = h = (b - a)/n$, $x_k = a + kh$, and $y_k = f(x_k)$.

**Note.** We can estimate the error involved in using the Trapezoid Rule to approximate a definite integral. If $f''$ is continuous and $M$ is any upper bound for the values of $|f''|$ on $[a, b]$, then

$$|E_T| = \left|\int_a^b f(x)\,dx - T\right| \leq \frac{b-a}{12}h^2M$$

where $h = (b - a)/n$.

**Note.** If $f(x) = mx + b$ then $f''(x) \equiv 0$ and $E_T = 0$. So the Trapezoid Rule gives exact values for such functions.

**Examples.** Page 381 number 8 I abc.

**Note.** Instead of approximating $y = f(x)$ with straight line segments, we can approximate it with parabolas. We then integrate to find the area under the parabolas. This leads to Simpson's Rule.

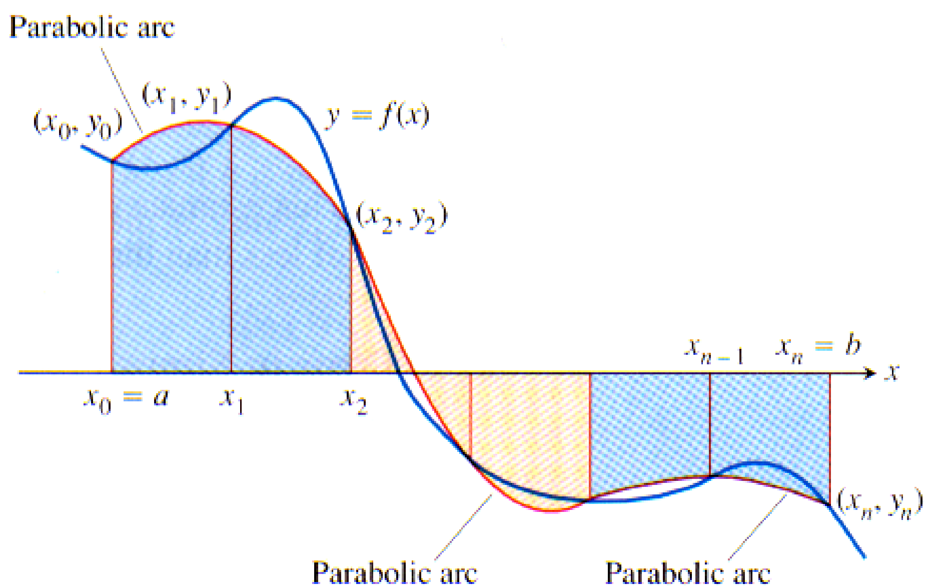


Figure 4.7.27, page 377

**Definition.** In *Simpson's Rule*, the integral $\int_a^b f(x)\,dx$, is approximated by

$$S = \frac{h}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n).$$

This approximation is based on a regular partition of $[a, b]$ of size $n$ where $n$ is even, and where $\Delta x_k = h = (b-a)/n$, $x_k = a + kh$, and $y_k = f(x_k)$.

**Note.** We can estimate the error involved in using Simpson's Rule to approximate a definite integral. If $f^{(4)}$ is continuous and $M$ is any upper bound for the values of $|f^{(4)}|$ on $[a, b]$, then

$$|E_S| = \left|\int_a^b f(x)\,dx - S\right| \leq \frac{b-a}{180}h^4 M$$

where $h = (b - a)/n$.

**Note.** If $f$ is a third degree polynomial then $f^{(4)}(x) \equiv 0$ and $E_S = 0$. So Simpson's Rule gives exact values for such functions.

**Examples.** Page 381 number 8 II abc, and page 382 number 16a.