Chapter 8. Infinite Series

8.1 Limits of Sequences of Numbers

Note. We now shift gears and consider a more theoretical, less tangible concept than those with which we have recently dealt.

Definition. An infinite *sequence* of numbers is a function whose domain is the set of integers greater than or equal to some integer n_0. We denote the sequence $\{f(n) \mid n \in \mathbb{N}, n \geq n_0\}$ as $\{a_n\}$ where $f(n) = a_n$.

Example. The *Fibonacci sequence* is defined recursively as: $a_1 = a_2 = 1$, $a_n = a_{n-1} + a_{n-2}$. The first few terms therefore are: 1, 1, 2, 3, 5, 8, 13, 21, 44, 65, ...

Definition. The sequence $\{a_n\}$ *converges* to the number L if for every $\epsilon > 0$ there exists an integer N such that for all $n > N$ we have $|a_n - L| < \epsilon$. If no such num,er L exists, then the seqeunce $\{a_n\}$ *diverges*. If $\{a_n\}$ converges to L, we write $\lim_{n\to\infty} a_n = L$ and call L the *limit* of the sequence.

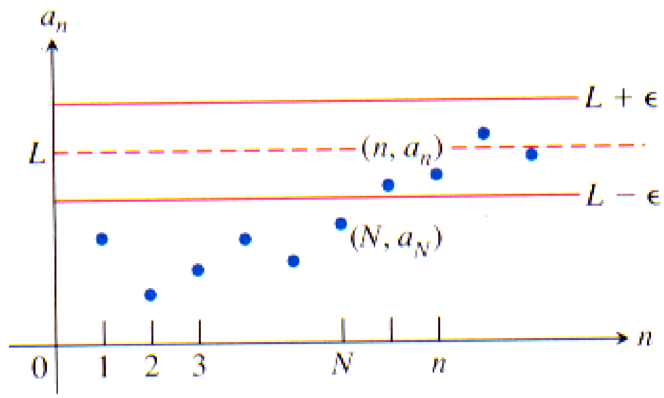

Figure 8.2 page 611

Example. Let $a_n = \dfrac{1}{n}$. Prove that $\lim_{n\to\infty} a_n = 0$.

Proof. Let $\epsilon > 0$ be given. Let N be an integer greater than $1/\epsilon$. Then for all $n > N$ we have $0 < a_n = 1/n < 1/N < \epsilon$, or $|a_n - 0| < \epsilon$. Therefore $\lim_{n\to\infty} a_n = 0$. *Q.E.D.*

Theorem 1. Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers and let A and B be real numbers. If $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$ then

1. *Sum Rule*: $\lim_{n\to\infty}(a_n + b_n) = A + B$.

2. *Difference Rule*: $\lim_{n\to\infty}(a_n - b_n) = A - B$.

3. *Product Rule*: $\lim_{n\to\infty} (a_n b_n) = AB$.

4. *Constant Multiple Rule*: $\lim_{n\to\infty} (kb_n) = kB$.

5. *Quotient Rule*: $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{A}{B}$ if $B \neq 0$.

Note. The proofs for each of these is similar to the proofs of the corresponding results for functions.

Example. Number 18 page 617.

Theorem 2. The Sandwich Theorem for Sequences.
Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences of real numbers. If $a_n \leq b_n \leq c_n$ holds for all n beyond some index N and is $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$ also.

Example. Number 22 page 617.

Theorem 3. Let $\{a_n\}$ be a sequence of real numbers. If $a_n \to L$ and if f is a function that is continuous at L and defined at all a_n, then $f(a_n) \to f(L)$.

Example. Number 24 page 617.

Theorem 4. Suppose that $f(x)$ is a function defined for all $x \geq n_0$ and that $\{a_n\}$ is a sequence of real numbers such that $a_n = f(n)$ for $n \geq n_0$. Then $\lim_{x\to\infty} f(x) = L$ implies that $\lim_{n\to\infty} a_n = L$.

Note. Theorem 4 allows us to use L'Hôpital's Rule on sequences.

Example. Number 67 page 618.

Note. We can use the method of the previous problem to verify the following:

1. $\lim_{n\to\infty} \dfrac{\ln n}{n} = 0$.

2. $\lim_{n\to\infty} \sqrt[n]{n} = 1$.

3. $\lim_{n\to\infty} x^{1/n} = 1$ for $x > 0$.

4. $\lim_{n\to\infty} x^n = 0$ for $|x| < 1$.

5. $\lim_{n\to\infty} \left(1 + \dfrac{x}{n}\right)^n = e^x$.

6. $\lim_{n\to\infty} \dfrac{x^n}{n!} = 0$.