Chapter 8. Infinite Series

8.2 Subsequences, Bounded Sequences, and Picard's Method

Definition. If the terms of one sequence appear in another sequence in their given order, we call the first sequence a *subsequence* of the second.

Definition. A sequence $\{a_n\}$ with the property that $a_n \leq a_{n+1}$ for all n is called a *nondecreasing sequence.* It is called *nonincreasing* if $a_n \geq a_{n+1}$ for all n. A sequence is *monotone* if it is either nondecreasing or nonincreasing.

Example. Number 14 page 626.

Definition. A sequence $\{a_n\}$ is *bounded from above* if there exists a number M such that $a_n \leq M$ for all n. The number M is an *upper bound* for $\{a_n\}$. The sequence is *bounded from below* if there exists a number m such that $m \leq a_n$ for all n. The number m is a *lower bound* for $\{a_n\}$. If it is bounded from above and below, then $\{a_n\}$ is a *bounded sequence.*

Theorem 5. Every bounded, monotonic sequence is convergent.

Example. Number 18 page 626.

Definition. A sequence can be defined recursively by giving:

1. The value(s) of the initial term or terms and

2. A rule, called a *recursion formula*, for calculating any later term from terms that precede it.

Example. Number 6 page 626.

Note. One common numerical method that leads to a sequence generated by recursion is called *Picard's method*. It allows us to find solutions of the equation $g(x) = x$ under certain conditions.

A theorem from advanced calculus tells us that if $g'(x)$ is continuous on a closed interval I whose interior contains a solution of the equation $g(x) = x$ and if $|g'(x)| < 1$ on I, then for any $x_0 \in I$ the sequence

$$x_0, x_1 = g(x_0), x_2 = g(x_1) = g(g(x_0)), \ldots, x_n = g(x_{n-1}) = g^n(x_0), \ldots$$

We illustrate this method with an example.

Example. Example 8 page 624: Solve the equation $\cos x = x$.

Solution. We know from the graphs that $y = \cos x$ and $y = x$ intersect for some $x \in (0, \pi/2)$. Notice that with $g(x) = \cos x$ we have $|g'(x)| = |-\sin x| = \sin x < 1$ for $x \in (0, \pi/2)$. Therefore by the above comment, we know we that we can find the solution by iterating g on any $x_0 \in (0, \pi/2)$. We choose $x_0 = 1$. We then get:

$$x_0 = 1, x_1 \approx 0.540, x_2 \approx 0.858, x_3 \approx 0.654, x_4 \approx 0.793,$$

$$x_5 \approx 0.701, x_6 \approx 0.764, x_7 \approx 0.722, x_8 \approx 0.750, x_9 \approx 0.731, \ldots$$

Continuing to iterate, we find $x \approx 0.739085133$. A picture of the dynamics is:

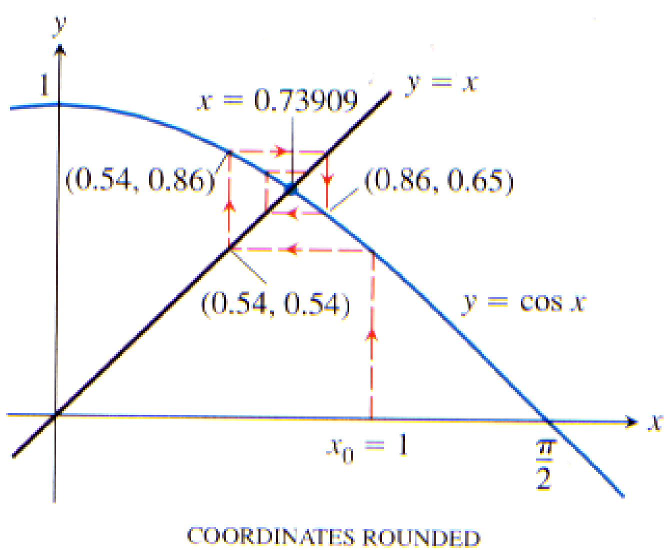

Figure 8.7 page 624

Examples. Numbers 30 and 26 page 626