Chapter 8. Infinite Series

8.6 Power Series

Definition. An expression of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots$$

is a *power series centered at* $x = 0$. An expression of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n + \cdots$$

is a *power series centered at* $x = a$. The term $c_n(x-a)^n$ is the n^{th} *term*, the number a is the *center*.

Note. We are interested in finding the values of x for which the above power series converge. By convention, a power series centered at a always converges to c_0 for $x = a$ (notice that the summation notation implies that we consider 0^0, but this is just a short-coming of the notation - the 0^{th} term is always c_0).

Example. We know that $\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots$ forms a geometric series and converges for $|x| < 1$ to $\dfrac{1}{1-x}$.

Example. For what x does $\displaystyle\sum_{n=0}^{\infty} \frac{3^n x^n}{n!}$ converge?

Solution. By the Ratio Test,

$$\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{3^{n+1}|x|^{n+1}}{(n+1)!} \frac{n!}{3^n|x|^n} = \lim_{n\to\infty} \frac{3|x|}{n+1} = 0.$$

Therefore this series converges absolutely for all x.

Theorem 12. There are three possibilities for $\displaystyle\sum_{n=0}^{\infty} c_n(x-a)^n$ with respect to convergence.

1. There is a positive number R such that the series diverges for $|x-a| > R$ but converges absolutely for $|x-a| < R$. The series may or may not converge at either of the endpoints $x = a - R$ and $x = a + R$.

2. The series converges absolutely for all x (that is, $R = \infty$).

3. The series converges at $x = a$ and diverges elsewhere (that is, $R = 0$).

Definition. The number R of Theorem 12 is the *radius of convergence*, and the set of all values of x for which the series converges is the *interval of convergence.*

Example. Example 3 page 663. Find the interval of convergence for $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{2n-1}}{2n-1}$.

Note. To find the interval of convergence for a power series:

1. Use the Ratio Test (or the Root Test) to find the interval where the series converges absolutely.

2. If the interval of absolute convergence is finite, test for convergence or divergence at each endpoint. Use a Comparison Test, the Integral Test, or the Alternating Series Test.

3. If the interval of absolute convergence is $a-R<x<a+R$, then the series diverges for $|x-a|>R$.

Example. Number 30 page 668.

Theorem 13. The Term-by-Term Differentiation Theorem
If $\sum_{n=0}^{\infty} c_n(x-a)^n$ converges for $a - R < x < a + R$ for some $R > 0$, it defines a function f:

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n, \;\; a - R < x < a + R.$$

Such a function has derivatives of all orders inside the interval of convergence and is said to be *analytic*. We can obtain the derivatives by differentiating the original series term by term:

$$\begin{aligned} f'(x) &= \sum_{n=1}^{\infty} nc_n(x-a)^{n-1} \\ f''(x) &= \sum_{n=2}^{\infty} n(n-1)c_n(x-a)^{n-2}, \end{aligned}$$

and so on. Each of these derived series converges at every interior point of the interval of convergence of the original series.

Note. The proof of Theorem 13 is found in an advanced calculus class or in an introductory analysis class (such as our MATH 4217/5217 Analysis). The following two theorems are other results from a more advanced class.

Theorem 14. The Term-by-Term Integration Theorem

Suppose that

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

converges for $a - R < x < a + R$ $(R > 0)$. Then

$$\sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$$

converges for $a - R < x < A + R$ and

$$\int f(x)\,dx = \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1} + C$$

for $a - R < x < a + R$.

Example. Number 42 page 669.

Theorem 15. The Series Multiplication Theorem for Power Series

If $A(x) = \sum_{n=0}^{\infty} a_n x^n$ and $B(x) = \sum_{n=0}^{\infty} b_n x^n$ converge absolutely for $|x| < R$ and if

$$c_n = a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \cdots + a_{n-1} b_1 + a_n b_0 = \sum_{k=0}^{n} a_k b_{n-k},$$

then $\sum_{n=0}^{\infty} c_n x^n$ converges absolutely to $A(x)B(x)$ for $|x| < R$:

$$\left(\sum_{n=0}^{\infty} a_n x^n \right) \cdot \left(\sum_{n=0}^{\infty} b_n x^n \right) = \sum_{n=0}^{\infty} c_n x^n.$$

Example. Example 7 page 667.