Physics 4617/5617: Quantum Physics
Course Lecture Notes

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Edition 5.1

Abstract

These class notes are designed for use of the instructor and students of the course Physics 4617/5617: Quantum Physics. This edition was last modified for the Fall 2006 semester.

VI. Angular Momentum and Spin

A. Angular Momentum.

1. In classical mechanics, the angular momentum of a particle (with respect to the origin) is

$$\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p}. \tag{VI-1}$$

Using the rule of cross-products, this gives three separate equations in Cartesian coordinates:

$$L_x = yp_z - zp_y, \quad L_y = zp_x - xp_z, \text{ and } L_z = xp_y - yp_x. \tag{VI-2}$$

2. Using Eq. (V-2), the corresponding quantum operators are:

$$L_x = \frac{\hbar}{i}\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right); \tag{VI-3}$$

$$L_y = \frac{\hbar}{i}\left(z\frac{\partial}{\partial y} - x\frac{\partial}{\partial z}\right); \tag{VI-4}$$

$$L_z = \frac{\hbar}{i}\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right). \tag{VI-5}$$

B. Eigenvalues of Angular Momentum.

1. We first ask the question, do the components of the angular momentum operator commute? To answer this question, let's introduce a test function $f(x, y, z)$ that the L_x and L_y operators will act upon. Then we can write

$$\begin{aligned}
[L_x, L_y]\, f &= \left(\frac{\hbar}{i}\right)^2 \left\{\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right) f \right.\\
&\quad \left. - \left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right)\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right) f\right\}\\
&= \left(\frac{\hbar}{i}\right)^2 \left\{y\frac{\partial}{\partial z}\left(z\frac{\partial f}{\partial x}\right) - y\frac{\partial}{\partial z}\left(x\frac{\partial f}{\partial z}\right)\right.
\end{aligned}$$

$$
\begin{aligned}
& -z\frac{\partial}{\partial y}\left(z\frac{\partial f}{\partial x}\right) + z\frac{\partial}{\partial y}\left(x\frac{\partial f}{\partial z}\right) - z\frac{\partial}{\partial x}\left(y\frac{\partial f}{\partial z}\right) \\
& +z\frac{\partial}{\partial x}\left(z\frac{\partial f}{\partial y}\right) + x\frac{\partial}{\partial z}\left(y\frac{\partial f}{\partial z}\right) - x\frac{\partial}{\partial z}\left(z\frac{\partial f}{\partial y}\right)\Big\} \\
= & \left(\frac{\hbar}{i}\right)^2 \left(y\frac{\partial f}{\partial x} + yz\frac{\partial^2 f}{\partial z\,\partial x} - yx\frac{\partial^2 f}{\partial z^2} - z^2\frac{\partial^2 f}{\partial y\,\partial x}\right. \\
& +zx\frac{\partial^2 f}{\partial y\,\partial z} - zy\frac{\partial^2 f}{\partial x\,\partial z} + z^2\frac{\partial^2 f}{\partial x\,\partial y} + xy\frac{\partial^2 f}{\partial z^2} \\
& \left. -x\frac{\partial f}{\partial y} - xz\frac{\partial^2 f}{\partial z\,\partial y}\right).
\end{aligned}
$$

2. All the terms cancel in pairs (by virtue of the equality of cross-derivatives) except two:

$$
[L_x, L_y]\, f = \left(\frac{\hbar}{i}\right)^2 \left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right) f = i\hbar L_z f,
$$

and we conclude (dropping the test function) that

$$
\boxed{[L_x, L_y] = i\hbar L_z.} \qquad \text{(VI-6)}
$$

3. By cyclic permutation of the indices, it follows that

$$
[L_y, L_z] = i\hbar L_x \quad \text{and} \quad [L_z, L_x] = i\hbar L_y. \qquad \text{(VI-7)}
$$

Exercise: Prove Eq. (VI-7) by the technique shown in §VI.B.1.

4. Since L_x, L_y, and L_z do not commute, they are *incompatible* observables. Hence, according to the generalized uncertainty principle (Eq. IV-100), we can write

$$
\sigma_{L_x}^2\, \sigma_{L_y}^2 \geq \left(\frac{1}{2i}\,\langle i\hbar L_z\rangle\right)^2 = \frac{\hbar^2}{4}\,\langle L_z\rangle^2,
$$

or

$$
\sigma_{L_x}\, \sigma_{L_y} \geq \frac{\hbar}{2}\,\left|\langle L_z\rangle\right|. \qquad \text{(VI-8)}
$$

5. It would therefore be futile to look for states that are simultaneously eigenfunctions of L_x and of L_y.

6. On the other hand, the *square* of the *total* angular momentum <u>does</u> commute with (separately) L_x, L_y, and L_z:

a) The square of the total angular momentum is

$$L^2 \equiv L_x^2 + L_y^2 + L_z^2. \tag{VI-9}$$

b) Using the definition of the commutator (Eq. IV-42) along with Eqs. (IV-43), (VI-6), (VI-7), and the following commutator relations,

$$\begin{aligned} \left[\hat{A}, \hat{B} + \hat{C}\right] &= \left[\hat{A}, \hat{B}\right] + \left[\hat{A}, \hat{C}\right] && \text{(VI-10)} \\ \left[\hat{A}, \hat{A}\right] &= 0, && \text{(VI-11)} \end{aligned}$$

we can write

$$\begin{aligned} \left[L^2, L_x\right] &= \left[L_x^2, L_x\right] + \left[L_y^2, L_x\right] + \left[L_z^2, L_x\right] \\ &= L_x\left[L_x, L_x\right] + \left[L_x, L_x\right]L_x + L_y\left[L_y, L_x\right] + \left[L_y, L_x\right]L_y \\ &\quad + L_z\left[L_z, L_x\right] + \left[L_z, L_x\right]L_z \\ &= 0 + 0 + L_y(-i\hbar L_z) + (-i\hbar L_z)L_y + L_z(i\hbar L_y) \\ &\quad + (i\hbar L_y)L_z \\ &= 0. \end{aligned}$$

Likewise, the same proof can be made for the L_y and L_z components giving

$$\left[L^2, L_x\right] = 0, \quad \left[L^2, L_y\right] = 0, \quad \left[L^2, L_z\right] = 0, \tag{VI-12}$$

or more compactly,

$$\left[L^2, \boldsymbol{L}\right] = 0. \tag{VI-13}$$

7. Since L^2 is compatible with each component of $\boldsymbol{L}$, we don't need to simultaneously find eigenstates for L^2 and each component of $\boldsymbol{L}$, instead only one component will suffice (that is to say that the eigenstates of the other two components will be identical to the one we work on).

a) Let us use L_z as our test component. Then we can write the following two eigenfunction equations:

$$L^2 f = \lambda f \quad \text{and} \quad L_z f = \mu f. \tag{VI-14}$$

b) We will now use a "ladder operator" technique similar to the one used for the *harmonic oscillator* (§III.E.5). Let

$$L_\pm \equiv L_x \pm i L_y. \tag{VI-15}$$

c) Its commutator with L_z is

$$\begin{aligned} [L_z, L_\pm] &= [L_z, L_x] \pm i\,[L_z, L_y] = i\hbar L_y \pm i(-i\hbar L_x) \\ &= \pm\hbar(L_x \pm i L_y), \end{aligned}$$

so

$$[L_z, L_\pm] = \pm\hbar L_\pm. \tag{VI-16}$$

d) Since each component of $\boldsymbol{L}$ is compatible with L^2 (as per Eq. VI-12), the following commutator results:

$$\left[L^2, L_\pm\right] = 0. \tag{VI-17}$$

Exercise: Prove Eq. (VI-17).

e) We can prove that $L_\pm f$ is an eigenfunction by making use of Eqs. (VI-14) and (VI-17), and substituting $L_\pm f$ for f, then

$$\begin{aligned} \left[L^2, L_\pm\right] f &= 0 \\ L^2(L_\pm f) - L_\pm(L^2 f) &= 0 \\ L^2(L_\pm f) &= L_\pm(L^2 f) \\ L^2(L_\pm f) &= L_\pm(\lambda f), \end{aligned}$$

and then from the first equation of Eq. (VI-14), we get

$$L^2(L_\pm f) = \lambda(L_\pm f), \qquad \text{(VI-18)}$$

so $L_\pm f$ is an eigenfunction of L^2 with the same eigenvalue λ.

8. Now let's take a look at the second eigenfunction equation of Eq. (VI-14). For this equation, let's replace f with $L_\pm f$ and write

$$\begin{aligned} L_z(L_\pm f) &= L_z L_\pm f - L_\pm L_z f + L_\pm L_z f \\ &= [L_z, L_\pm]\, f + L_\pm(L_z f) \\ &= \pm\hbar L_\pm f + L_\pm(\mu f) \\ &= (\mu \pm \hbar)(L_\pm f), \qquad \text{(VI-19)} \end{aligned}$$

so $L_\pm f$ is an eigenfunction of L_z with the new eigenvalues $\mu \pm \hbar$.

9. L_+ is called the "**raising**" operator because it increases the eigenvalue of L_z by $\hbar$.

10. L_- is called the "**lowering**" operator because it decreases the eigenvalue of L_z by $\hbar$.

11. We also can investigate various relationships with these angular momentum ladder operators.

$$\begin{aligned} L_+ L_- &= (L_x + iL_y)(L_x - iL_y) \\ &= L_x^2 - iL_xL_y + iL_yL_x + L_y^2 \\ &= L_x^2 + L_y^2 - i[L_x, L_y], \qquad \text{(VI-20)} \end{aligned}$$

$$\begin{aligned} L_- L_+ &= (L_x - iL_y)(L_x + iL_y) \\ &= L_x^2 + iL_xL_y - iL_yL_x + L_y^2 \\ &= L_x^2 + L_y^2 + i[L_x, L_y], \qquad \text{(VI-21)} \end{aligned}$$

and the commutator of L_+ and L_- results from subtracting Eq. (VI-21) from Eq. (VI-20) which gives

$$[L_+, L_-] = L_+L_- - L_-L_+ = -2i[L_x, L_y] = 2\hbar L_z. \tag{VI-22}$$

Exercise: Verify the following commutator relations:

$$\begin{aligned} [L_z, L_+] &= \hbar L_+ && \text{(VI-23)} \\ [L_z, L_-] &= -\hbar L_- && \text{(VI-24)} \\ \left[L^2, L_+\right] &= \left[L^2, L_-\right] = 0. && \text{(VI-25)} \end{aligned}$$

If we add Eqs. (VI-20) and (VI-21) we get

$$L_+L_- + L_-L_+ = 2(L_x^2 + L_y^2) = 2(L^2 - L_z^2). \tag{VI-26}$$

Finally, if we use Eq. (VI-26) along with Eq. (VI-9) we get

$$\begin{aligned} L^2 &= L_x^2 + L_y^2 + L_z^2 \\ &= \frac{1}{2}(L_+L_- + L_-L_+) + L_z^2 \\ &= L_+L_- - \hbar L_z + L_z^2 && \text{(VI-27)} \\ &= L_-L_+ + \hbar L_z + L_z^2. && \text{(VI-28)} \end{aligned}$$

Exercise: Prove Eqs. (VI-27) and (VI-28).

12. For a given value of λ in Eq. (VI-14), a "ladder" of states is obtained with each "rung" separated from its neighbors by one unit of $\hbar$ in the eigenvalue of L_z (see Figure VI-1).

a) To ascend the ladder, we apply the raising operator.

b) To descend the ladder, we apply the lowering operator.

c) This procedure cannot go on forever — eventually a state is reached for which the z-component exceeds the *total* angular momentum which is impossible (see below).

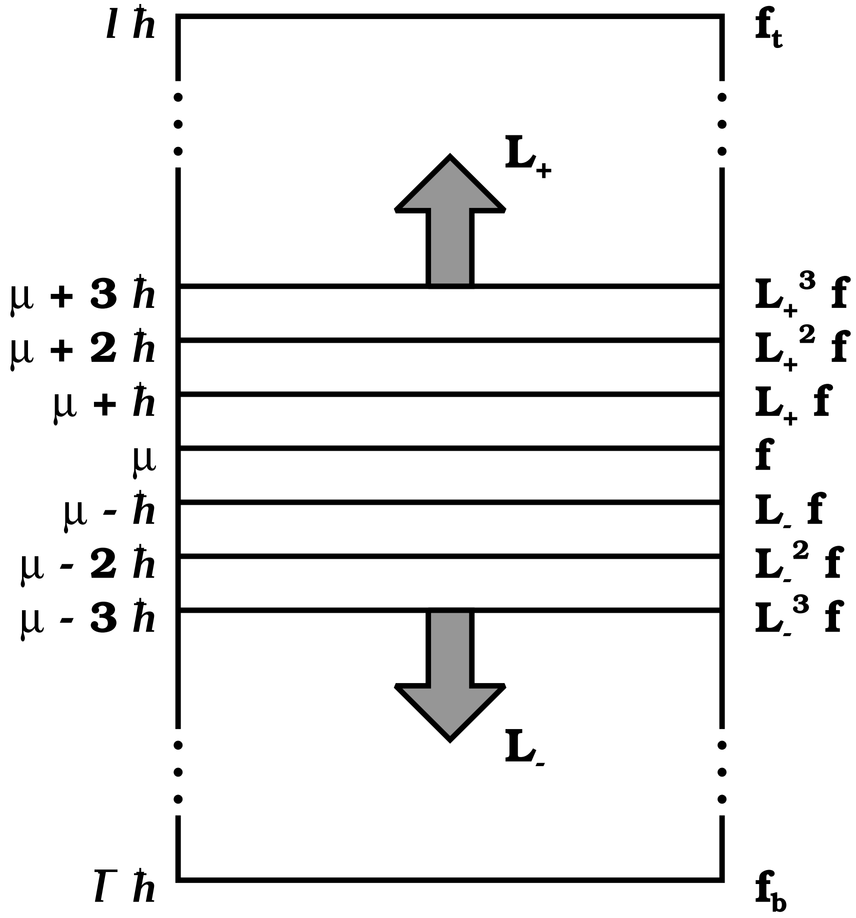

Figure VI–1: **The "ladder" angular momentum states.**

13. As such, there must exist a "top rung," f_t, such that

$$L_+ f_t = 0. \tag{VI-29}$$

Let $\ell\hbar$ be the eigenvalue of L_z at this <u>top</u> rung.

$$L_z f_t = \ell\hbar f_t; \quad L^2 f_t = \lambda f_t. \tag{VI-30}$$

a) Using our original eigenvalue equations (*i.e.*, Eq. VI-14), we can write

$$(L_x^2 + L_y^2) f_t = (L^2 - L_z^2) f_t = (\lambda - \mu^2) f_t. \tag{VI-31}$$

b) Since L_x, L_y, L_z, and L^2 are Hermitian operators with real eigenvalues, the square of these eigenvalues must be *positive* numbers. Thus the operator $L_x^2 + L_y^2$ must have a positive eigenvalue and $\lambda \geq \mu^2 = \ell^2\hbar^2$, hence supporting the statement made in §VI.B.12.c.

c) Using Eq. (VI-29), it is obvious that

$$L_- L_+ f_t = L_-(L_+ f_t) = L_-(0) = 0. \tag{VI-32}$$

d) Now, using the second equation of Eq. (VI-30) in conjunction with the first equation of Eq. (VI-30) along with Eq. (VI-28), we can write

$$\begin{aligned} L^2 f_t &= (L_- L_+ + \hbar L_z + L_z^2) f_t = (0 + \ell\hbar^2 + \ell^2\hbar^2) f_t \\ &= (\ell\hbar^2 + \ell^2\hbar^2) f_t = \ell(\ell+1)\hbar^2 f_t = \lambda f_t \end{aligned} \tag{VI-33}$$

14. Similarly, there must exist a "bottom rung," f_b, such that

$$L_- f_b = 0. \tag{VI-34}$$

Let $\overline{\ell}\hbar$ be the eigenvalue of L_z at this bottom rung, then

$$L_z f_b = \overline{\ell}\hbar f_b; \quad L^2 f_b = \lambda f_b. \tag{VI-35}$$

a) Using Eq. (VI-34), we see that

$$L_+ L_- f_b = L_+(L_- f_b) = L_+(0) = 0. \tag{VI-36}$$

b) Now, using the second equation of Eq. (VI-35) in conjunction with the first equation of Eq. (VI-35) along with Eq. (VI-27), we can write

$$\begin{aligned} L^2 f_b &= (L_+ L_- - \hbar L_z + L_z^2) f_b = (0 - \overline{\ell}\hbar^2 + \overline{\ell}^2\hbar^2) f_b \\ &= (-\overline{\ell}\hbar^2 + \overline{\ell}^2\hbar^2) f_b = \overline{\ell}(\overline{\ell}-1)\hbar^2 f_b = \lambda f_b \end{aligned} \tag{VI-37}$$

15. It is clear from the comparison of Eq. (VI-33) with Eq. (VI-37) that

$$\frac{\lambda}{\hbar^2} = \ell(\ell+1) = \overline{\ell}(\overline{\ell}-1). \tag{VI-38}$$

a) This implies that either $\overline{\ell} = -\ell$ or $\overline{\ell} = \ell + 1$. We can reject the second solution since ℓ is defined to be the top most rung and this second equation would place $\overline{\ell}$ above ℓ.

b) Let N be the number of *steps* one must make from the bottom ($\overline{\ell}$) state to get to the top (ℓ) state, then

$$N = \ell - \overline{\ell} = \ell + \ell,$$

or

$$\ell = \frac{N}{2}. \qquad \text{(VI-39)}$$

c) At this point, let's note that L^2 (the angular momentum squared) has the same units as $\hbar$ squared (remember that ℓ is either an integer or half-integer as shown in Eq. VI-39), hence the angular momentum has units of $\hbar$. As such, let's rewrite the eigenvalue of the L_z operator as $\mu = m\hbar$.

d) From these last two points, we see that the maximum value of m is ℓ (see Eq. VI-30) and the minimum value of m is $-\ell$ (see Eq. VI-35), where ℓ can be an integer or half-integer.

e) From this formalism, it is clear that there are $2\ell + 1$ "rungs" (*i.e.*, values of m) in the angular momentum ladder: $\ell, \ell-1, \ell-2, ..., 1, 0, -1, ..., -\ell$ as shown in Figure (VI-2).

16. Since the eigenvalues are composed of ℓ's and m's, so too must be the eigenfunctions. As such, we can now write the two angular momentum eigenvalue equations as

$$\boxed{L^2 f_\ell^m = \ell(\ell+1)\hbar^2 f_\ell^m; \quad L_z f_\ell^m = m\hbar f_\ell^m\ ,} \qquad \text{(VI-40)}$$

where

$$\ell = 0, \frac{1}{2}, 1, \frac{3}{2}, ...; \quad m = -\ell, -\ell+1, ..., \ell-1, \ell. \qquad \text{(VI-41)}$$

This is the reason why we chose $\ell(\ell+1)$ as our differential equation constant for the Schrödinger equation in Eqs. (V-14) and (V-15).

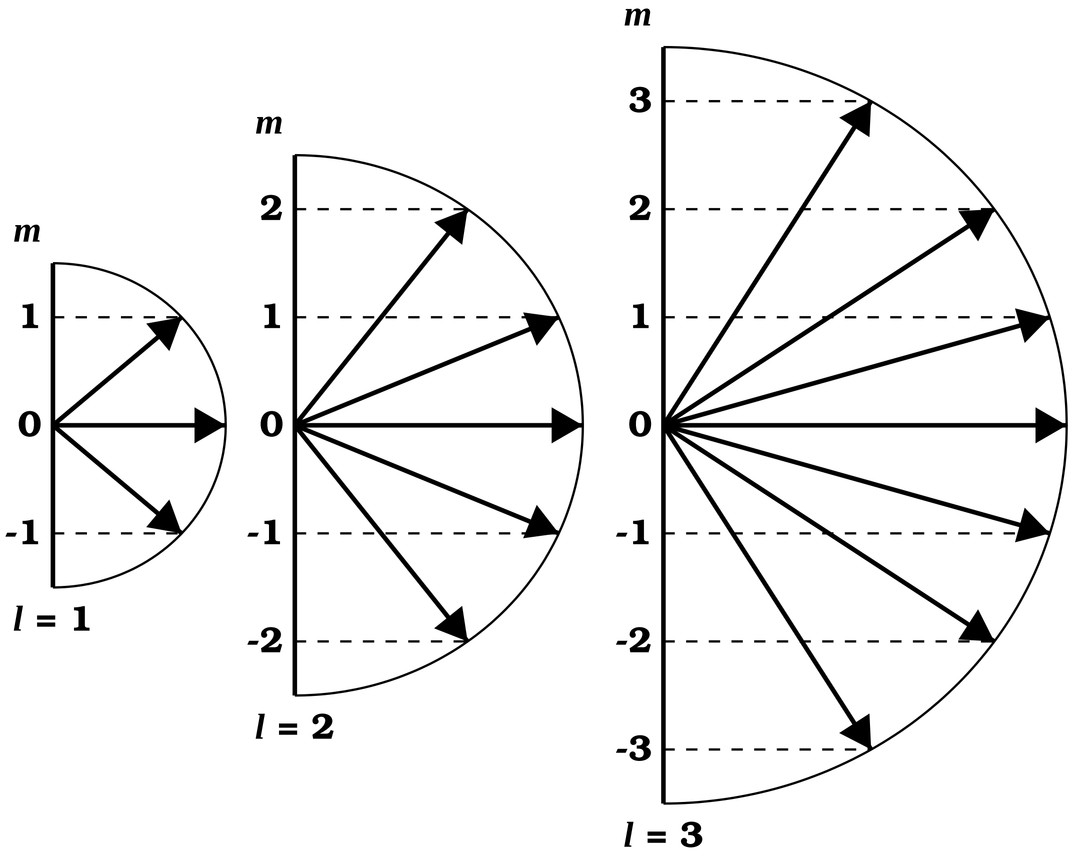

Figure VI–2: **The allowed projections of the angular momentum for the cases of $\ell = 1$, 2, and 3.**

Example VI–1. The angular momentum operators:

(a) Prove that if f is simultaneously an eigenfunction of L^2 and L_z, the square of the eigenvalue of L_z cannot exceed the eigenvalue of L^2.

Using Eq. (VI-14) we have

$$\langle L^2 \rangle = \langle f | L^2 | f \rangle = \langle f | L^2 f \rangle = \langle f | \lambda f \rangle = \lambda \langle f | f \rangle = \lambda$$

and

$$\langle L^2 \rangle = \langle L_x^2 + L_y^2 + L_z^2 \rangle = \langle L_x^2 \rangle + \langle L_y^2 \rangle + \mu^2 .$$

But L_x is Hermitian, so

$$\langle L_x^2 \rangle = \langle f | L_x^2 f \rangle = \langle L_x f | L_x f \rangle = || L_x f ||^2 \geq 0 ,$$

and likewise $L_y \geq 0$. So

$$\lambda \geq \mu^2 \qquad \text{Q.E.D.}$$

(b) As it turns out (see Equations VI-40 and VI-41), the square of the eigenvalue of L_z never even *equals* the eigenvalue of L^2 (except in the special case $\ell = m = 0$). Comment on the implications of this result. Show that it is enforced by the uncertainty principle (see Eq. VI-8), and explain how the special case gets away with it.

Implication : There is no state in which the angular momentum points in a specific direction. If there were, then we would simultaneously know L_x, L_y, and L_z — but these are incompatible observables — if you know L_x precisely, you cannot know L_y and L_z precisely. The uncertainty principle of Eq. (VI-8) says that L_x, L_y, and L_z cannot be simultaneously determined, except in the special case $\langle L_x \rangle = \langle L_y \rangle = \langle L_z \rangle = 0$, which is to say when $\ell = 0$ and $m = 0$.

C. Eigenfunctions of Angular Momentum.

1. We need to rewrite L_x, L_y, and L_z (see Eqs. VI-3, 4, 5) in spherical coordinates.

a) The angular momentum operator in vector notation is

$$\boldsymbol{L} = \frac{\hbar}{i}\,(\boldsymbol{r} \times \boldsymbol{\nabla}), \qquad \text{(VI-42)}$$

where the gradient operator in spherical coordinates is

$$\boldsymbol{\nabla} = \hat{r}\,\frac{\partial}{\partial r} + \hat{\theta}\,\frac{1}{r}\,\frac{\partial}{\partial \theta} + \hat{\phi}\,\frac{1}{r\,\sin\theta}\,\frac{\partial}{\partial \phi}, \qquad \text{(VI-43)}$$

and $\boldsymbol{r} = r\hat{r}$, so

$$\boldsymbol{L} = \frac{\hbar}{i}\left[r(\hat{r} \times \hat{r})\,\frac{\partial}{\partial r} + (\hat{r} \times \hat{\theta})\,\frac{\partial}{\partial \theta} + (\hat{r} \times \hat{\phi})\,\frac{1}{\sin\theta}\,\frac{\partial}{\partial \phi}\right]. \qquad \text{(VI-44)}$$

But $(\hat{r} \times \hat{r}) = 0, (\hat{r} \times \hat{\theta}) = \hat{\phi}$, and $(\hat{r} \times \hat{\phi}) = -\hat{\theta}$, and hence

$$\boldsymbol{L} = \frac{\hbar}{i} \left(\hat{\phi} \frac{\partial}{\partial \theta} - \hat{\theta} \frac{1}{\sin\theta} \frac{\partial}{\partial \phi} \right). \qquad \text{(VI-45)}$$

b) We now need to express the unit vectors $\hat{\theta}$ and $\hat{\phi}$ in their Cartesian components:

$$\begin{aligned} \hat{\theta} &= (\cos\theta\, \cos\phi)\, \hat{x} + (\cos\theta\, \sin\phi)\, \hat{y} - (\sin\theta)\, \hat{z} \\ \hat{\phi} &= -(\sin\phi)\, \hat{x} + (\cos\phi)\, \hat{y}. \end{aligned}$$

c) Using these unit vector relations in Eq. (VI-45) gives us

$$\begin{aligned} \boldsymbol{L} &= \frac{\hbar}{i} \left[(-\sin\phi\, \hat{x} + \cos\phi\, \hat{y}) \frac{\partial}{\partial \theta} \right. && \text{(VI-46)} \\ & \left. - \, (\cos\theta\, \cos\phi\, \hat{x} + \cos\theta\, \sin\phi\, \hat{y} - \sin\theta\, \hat{z}) \frac{1}{\sin\theta} \frac{\partial}{\partial \phi} \right]. \end{aligned}$$

Looking at the Cartesian unit vector components for comparison with $\boldsymbol{L} = L_x\, \hat{x} + L_y\, \hat{y} + L_z\, \hat{z}$, we see that

$$\begin{aligned} L_x &= \frac{\hbar}{i} \left(-\sin\phi \frac{\partial}{\partial \theta} - \cos\phi\, \cot\theta \frac{\partial}{\partial \phi} \right), && \text{(VI-47)} \\ L_y &= \frac{\hbar}{i} \left(+\cos\phi \frac{\partial}{\partial \theta} - \sin\phi\, \cot\theta \frac{\partial}{\partial \phi} \right), && \text{(VI-48)} \end{aligned}$$

and

$$L_z = \frac{\hbar}{i} \frac{\partial}{\partial \phi}. \qquad \text{(VI-49)}$$

d) We shall also need the raising and lowering operators:

$$\begin{aligned} L_{\pm} &= L_x \pm i L_y \\ &= \frac{\hbar}{i} \left[(-\sin\phi \pm i \cos\phi) \frac{\partial}{\partial \theta} - (\cos\phi \pm i \sin\phi)\, \cot\theta \frac{\partial}{\partial \phi} \right], \end{aligned}$$

but $\cos\phi \pm i \sin\phi = e^{\pm i\phi}$, so handling the raising and lowering operators separately we have

$$\begin{aligned} L_{+} &= \hbar \left[(+\cos\phi + i \sin\phi) \frac{\partial}{\partial \theta} + i(\cos\phi + i \sin\phi)\, \cot\theta \frac{\partial}{\partial \phi} \right] \\ &= +\hbar e^{+i\phi} \left(\frac{\partial}{\partial \theta} + i \cot\theta \frac{\partial}{\partial \phi} \right) \end{aligned}$$

and

$$\begin{aligned}
L_- &= \hbar\left[(-\cos\phi + i\sin\phi)\,\frac{\partial}{\partial\theta} + i(\cos\phi - i\sin\phi)\,\cot\theta\,\frac{\partial}{\partial\phi}\right] \\
&= -\hbar\left[(\cos\phi - i\sin\phi)\,\frac{\partial}{\partial\theta} - i(\cos\phi - i\sin\phi)\,\cot\theta\,\frac{\partial}{\partial\phi}\right] \\
&= -\hbar e^{-i\phi}\left(\frac{\partial}{\partial\theta} - i\cot\theta\,\frac{\partial}{\partial\phi}\right),
\end{aligned}$$

so

$$L_\pm = \pm\hbar e^{\pm i\phi}\left(\frac{\partial}{\partial\theta} \pm i\cot\theta\,\frac{\partial}{\partial\phi}\right). \qquad \text{(VI-50)}$$

e) We are now in a position to determine $f_\ell^m(\theta,\phi)$. It's an eigenfunction of L_z with an eigenvalue of $m\hbar$:

$$L_z f_\ell^m = \frac{\hbar}{i}\,\frac{\partial f_\ell^m}{\partial\phi} = m\hbar f_\ell^m,$$

i) The solution to this simple differential equation is

$$f_\ell^m = g(\theta)\,e^{im\phi}. \qquad \text{(VI-51)}$$

ii) The function $g(\theta)$ is a constant of integration in ϕ, but it can still depend upon θ.

iii) Remember from Eq. (V-21) that $\Phi(\phi) = e^{im\phi}$ for the ϕ component of the 3-D Schrödinger equation, hence

$$f_\ell^m = g(\theta)\,\Phi(\phi). \qquad \text{(VI-52)}$$

f) We will now retrieve the functional form of $g(\theta)$ by making use of our second eigenvalue equation for the angular momentum operator. Using the first equation of Eq. (VI-40) in conjunction with Eq. (VI-27), we get

$$
\begin{aligned}
L^2 f_\ell^m &= \left(L_+L_- + L_z^2 - \hbar L_z\right) f_\ell^m \\
&= \hbar e^{i\phi}\left(\frac{\partial f_\ell^m}{\partial\theta} + i\cot\theta\,\frac{\partial f_\ell^m}{\partial\phi}\right)\left(-\hbar e^{-i\phi}\right) \\
&\quad \times\left(\frac{\partial f_\ell^m}{\partial\theta} - i\cot\theta\,\frac{\partial f_\ell^m}{\partial\phi}\right) - \hbar^2\frac{\partial^2 f_\ell^m}{\partial\phi^2} - \frac{\hbar^2}{i}\,\frac{\partial f_\ell^m}{\partial\phi} \\
&= \hbar^2\,\ell(\ell+1)\,f_\ell^m. \qquad \text{(VI-53)}
\end{aligned}
$$

i) At this point, let's make use of Eq. (VI-51) in Eq. (VI-53). Taking the θ and ϕ derivatives of Eq. (VI-51), we get:

$$
\begin{aligned}
\frac{\partial f_\ell^m}{\partial\theta} &= e^{im\phi}\,\frac{dg}{d\theta} \qquad \text{(VI-54)} \\
\frac{\partial f_\ell^m}{\partial\phi} &= ime^{im\phi}\,g. \qquad \text{(VI-55)}
\end{aligned}
$$

ii) Using these equations in Eq. (VI-53) gives

$$
\begin{aligned}
&-e^{i\phi}\left(\frac{\partial}{\partial\theta} + i\,\cot\theta\,\frac{\partial}{\partial\phi}\right)\left(e^{i(m-1)\phi}\right)\left(\frac{\partial g}{\partial\theta} + mg\,\cot\theta\right) \\
&\quad + m^2 g e^{im\phi} - mg e^{im\phi} \\
&= e^{im\phi}\left[-\frac{d}{d\theta}\left(\frac{dg}{d\theta} + mg\cot\theta\right) + (m-1)\,\cot\theta\,\left(\frac{dg}{d\theta} + mg\cot\theta\right)\right. \\
&\quad \left. + m(m-1)g\right] = \ell(\ell+1)g e^{im\phi}.
\end{aligned}
$$

iii) Canceling $e^{im\phi}$ gives

$$
\begin{aligned}
&-\frac{d^2g}{d\theta^2} - m\,\frac{dg}{d\theta}\,\cot\theta + mg\,\csc^2\theta + (m-1)\,\cot\theta\,\frac{dg}{d\theta} \\
&\quad + m(m-1)\left(1+\cot^2\theta\right)g \\
&= -\frac{d^2g}{d\theta^2} - \cot\theta\,\frac{dg}{d\theta} + m^2 g\,\csc^2\theta = \ell(\ell+1)g.
\end{aligned}
$$

iv) Now, multiplying each term by $-\sin\theta$ produces

$$\sin^2\theta\,\frac{d^2g}{d\theta^2}+\sin\theta\,\cos\theta\,\frac{dg}{d\theta}-m^2g=-\ell(\ell+1)g\,\sin^2\theta.$$

v) This is a differential equation for $g(\theta)$, but we can write it in the more familiar form of

$$\sin\theta\,\frac{d}{d\theta}\left(\sin\theta\,\frac{dg}{d\theta}\right)+[\ell(\ell+1)\,\sin^2\theta-m^2]\,g=0. \tag{VI-56}$$

2. Note that Eq. (VI-56) is precisely the equation we determined for the θ component of the three dimensional Schrödinger equation (see Eq. V-24): $g(\theta)=\Theta(\theta)$. Using this realization in Eq. (VI-52) means that

$$f_\ell^m=\Theta(\theta)\,\Phi(\phi)=Y_\ell^m. \tag{VI-57}$$

a) As such, ***spherical harmonics are precisely the* (normalized) *eigenfunctions of* L^2 *and* L_z.**

b) When we solved the Schrödinger equation by separation of variables in §V.A of the notes, we were inadvertently constructing simultaneous eigenfunctions of the three commuting operators H, L^2, and L_z:

$$H\psi=E\psi,\quad L^2\psi=\ell(\ell+1)\hbar^2\psi,\quad L_z\psi=m\hbar\psi. \tag{VI-58}$$

c) There is a slight inconsistency here:

i) The *algebraic* theory of angular momentum permits ℓ (and hence also m) to take on *half*-integer values (see Eq. VI-41), whereas the *analytic* method yielded eigenfunctions only for *integer* values (see Eq. V-28).

ii) So, does this mean that the half-integer solutions are spurious? The answer is no, these half-integer solutions are of profound importance as will be seen on the next section concerning "spin." They can, however, be ignored for the description of the orbital angular momentum.

Example VI–2. This example incorporates quantum mechanics in classical mechanics. Assume, two particles of mass M are attached to the ends of a massless rigid rod of length a. The system is free to rotate in three dimensions about the center of the rod (but the center point itself is fixed). This technique is actually used to describe rotational transitions in diatomic molecules.

(a) Show the allowed energies of this **rigid rotor** are

$$E_n = \frac{\hbar^2 n(n+1)}{Ma^2}, \quad \text{for} \quad n = 0, 1, 2, ...$$

Hint: First express the (classical) energy in terms of the total angular momentum.

Classically, the energy of each mass is just going to be the kinetic energy of the mass (potential energy is zero in this problem). Even though the masses are revolving around a given point, we will just use the standard definition of kinetic energy here: $(1/2)mv^2$. Then, the Hamiltonian (or the total energy) will be

$$H = 2\left(\frac{1}{2}mv^2\right) = mv^2.$$

Now, the angular momentum of a mass revolving around an axis is $L = Mrv$, where $r = a/2$ is the distance that the mass is from the axis. Since there are two masses, we have

$$L = 2\frac{a}{2}Mv,$$

so

$$L^2 = a^2M^2v^2 \quad \text{and thus} \quad H = \frac{L^2}{Ma^2}.$$

But from quantum mechanics, we know the eigenvalues of L^2 are $\hbar^2\ell(\ell+1)$, or, since we usually label energies with n and E is the eigenvalue for H, we get

$$\boxed{E_n = \frac{\hbar^2 n(n+1)}{Ma^2} \qquad (n = 0, 1, 2, ...).}$$

(b) What are the normalized eigenfunctions for this system? What is the degeneracy of the nth energy level?

Since there is no radial component to this problem, the eigenfunction is just going to be described by spherical harmonics:

$$\boxed{\psi_{nm}(\theta,\phi) = Y_n^m(\theta,\phi)\ .}$$

The degeneracy of the nth energy level is just the number of m-values for a given n which is given in §VI.B.15.e as $\boxed{2n+1}$.

D. Spin and Particle Families

1. In *classical* mechanics, a rigid body admits two kinds of angular momentum:

a) **Orbital**: $\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p}$, associated with the motion <u>*of*</u> the center of mass. Such motion is referred to as a **revolution** about the center of mass.

b) **Spin**: $\boldsymbol{S} = I\boldsymbol{\omega}$, associated with the motion <u>*about*</u> the center of mass. Such motion is referred to as a **rotation** about an axis.

c) One then talks about the **<u>total</u> angular momentum**: orbital ($\boldsymbol{L}$) + spin ($\boldsymbol{S}$) angular momenta.

2. By analogy, we have the same description on the microscopic level for *quantum* mechanics:

a) **Orbital**: The motion of an electron about the nucleus of an atom as described by spherical harmonics with the orbital angular momentum quantum number ℓ and the magnetic (or azimuthal) quantum number m. This is sometimes referred to as the *extrinsic* angular momentum ($\boldsymbol{L}$).

b) **Spin**: Unlike the classical case, this isn't the spin of the electron about an axis, since the electron is a point particle (even though we describe electron spin about an axis in elementary physics). Here "spin" is nothing more than the *intrinsic* angular momentum ($\boldsymbol{S}$) of the electron. Since this is intrinsic, spin angular momentum is independent of spatial coordinates (r, θ, ϕ).

3. In quantum mechanics, the *algebraic* theory of spin is just a carbon copy of the theory of orbital angular momentum.

a) It obeys the following fundamental commutation relations:

$$[S_x, S_y] = i\hbar S_z, \quad [S_y, S_z] = i\hbar S_x, \quad [S_z, S_x] = i\hbar S_y. \tag{VI-59}$$

b) The eigenvectors S^2 and S_z satisfy:

$$S^2|s\,m_s\rangle = s(s+1)\hbar^2|s\,m_s\rangle; \quad S_z|s\,m_s\rangle = m_s\hbar|s\,m_s\rangle. \tag{VI-60}$$

Note that the L^2 and L_z eigenstates are <u>eigenfunctions</u> whereas the S^2 and S_z eigenstates are <u>eigenvectors</u>. As such, Eq. (VI-60) uses the “ket” notation for the eigenstate. Note, too, that we could have used this ket notation for the orbital angular momentum eigenstate with the realization that $|\ell\, m\rangle \equiv Y_\ell^m$.

c) We also can introduce ladder operators $S_\pm \equiv S_x \pm iS_y$ where we have

$$L_\pm Y_\ell^m = L_\pm|\ell\, m\rangle = \hbar\sqrt{\ell(\ell+1) - m(m\pm 1)}\,|\ell\,(m\pm 1)\rangle \tag{VI-61}$$

for the orbital angular momentum ladder operators and

$$S_\pm|s\, m_s\rangle = \hbar\sqrt{s(s+1) - m_s(m_s\pm 1)}\,|s\,(m_s\pm 1)\rangle \tag{VI-62}$$

for the spin angular momentum ladder operators.

Exercise: Prove Eq. (VI-61) by making use of the fact L^2 and L_z are Hermitian and that $|\ell\, m\rangle$ must be normalizable.

4. The spin angular momentum eigenvectors are *not* spherical harmonics (they are not functions of θ and ϕ at all). As such, there is no reason to exclude the half-integer values of s and m_s:

$$s = 0, \frac{1}{2}, 1, \frac{3}{2}, ...; \quad m_s = -s, -s+1, ..., s-1, s. \tag{VI-63}$$

a) It so happens that every elementary particle has a *specific and immutable* value of s which we call the **spin** of that particular species.

b) Particles with *half-integer* spins are called **fermions**. Fermions are said to have **antisymmetrical** wave functions.

c) Particles with *integer* spins are called **bosons**. Bosons are said to have **symmetrical** wave functions.

d) The convolution (*i.e.*, joining together) of antisymmetrical wave functions go to zero as two identical particles approach each other. As a result, two fermions in the same quantum state exhibit mutual repulsion to avoid their combined wave functions going to zero. This effect is known as the **Pauli Exclusion Principle**.

e) Bosons have no such exclusion principle since the convolution of symmetric wave functions do not go to zero as two identical particles approach each other.

5. Before going on further, let's discuss particle physics. There are two main groups of particles that make up all matter and energy:

a) **Elementary particles**: These are particles that make up matter. They are subdivided into 3 groups:

i) **Leptons** (*light* particles) include the *electron* (e^-, $m_e = 511$ keV, 1 keV = 1000 eV, 1 eV $= 1.60 \times 10^{-19}$ Joules), *muon* (μ, $m_\mu = 107$ MeV), and *tau particle* (τ, $m_\tau = 1784$ MeV), each with a negative charge; their respective neutrinos: *electron neutrino* (ν_e, $m_{\nu-e} < 30$ eV), *muon neutrino* (ν_μ, $m_{\nu-\mu} < 0.5$ MeV), and *tau neutrino* (ν_τ, $m_{\nu-\tau} < 250$ MeV), each with no charge; and the antiparticles of each: e^+ (called a *positron*), $\overline{\mu}, \overline{\tau}, \overline{\nu_e}, \overline{\nu_\mu}$, and $\overline{\nu_\tau}$. These particles do <u>**not**</u> participate in the strong interactions. All leptons have spin of 1/2, hence are fermions.

ii) **Mesons** are particles of intermediate mass that are made of quark-antiquark pairs and include *pions*, *kaons*, and *η-particles*. All are unstable and decay via weak or E/M interactions. All mesons have either 0 or integer spin, hence are bosons.

    - **iii) Baryons** (*heavy* particles) include the nucleons n (*neutrons* — neutral particles) and p (*protons* — positive charged) and the more massive *hyperons* (*i.e.*, Λ, Σ, Ξ, and Ω). Baryons are composed of a triplet of quarks. Each baryon has an antibaryon associated with it and has a spin of either 1/2 or 3/2, hence are fermions.

- **b) Field particles**: These particles mediate the 4 natural forces and are sometimes referred to as the *energy* particles — all are bosons. Here they are listed in order from strongest to weakest:
    - **i) Gluons**: Mediate the strong [nuclear] force. Strength of this force is described by the **color charge**.
    - **ii) Photons**: Mediate the electromagnetic force. Strength of this force is described by the **electric charge**.
    - **iii) Intermediate vector [W & Z] bosons** (sometimes referred to as **weakons**): Mediate the weak [nuclear] force. Strength of this force is described by the **weak charge**.
    - **iv) Gravitons**: Mediate the force of gravity. Strength of this force is described by the **mass**.

6. From the above list of elementary particles, there seems to be only 2 types of basic particles: *leptons* which do not obey the strong force and *quarks* which do obey the strong force. Particles that participate in the strong force are also called **hadrons** — so obviously, the hadrons are those particles composed of quarks (*i.e.*, the mesons and baryons). All quarks have a spin of 1/2.

There are 6 *flavors* of leptons (as describe above). As such, it was theorized and later observed, 6 flavors or *colors* of quarks (and an additional 6 antiquarks) must exist:

a) **Up** (u) quark has a rest energy of 360 MeV (1 MeV = 10^6 eV) and a charge of $+\frac{2}{3}\,e$.

b) **Down** (d) quark has a rest energy of 360 MeV and a charge of $-\frac{1}{3}\,e$.

c) **Charmed** (c) quark has a rest energy of 1500 MeV and a charge of $+\frac{2}{3}\,e$.

d) **Strange** (s) quark has a rest energy of 540 MeV and a charge of $-\frac{1}{3}\,e$.

e) **Top** (t) quark has a rest energy of 170 GeV (1 GeV = 10^9 eV) and a charge of $+\frac{2}{3}\,e$.

f) **Bottom** (b) quark has a rest energy of 5 GeV and a charge of $-\frac{1}{3}\,e$.

7. Note that a proton is composed of 2 u and a d quark and a neutron composed of an u and 2 d quarks.

8. The theory on how quarks interact with each other is called **quantum chromodynamics** (in analogy with quantum mechanics). One interesting result of this theory is that quarks cannot exist in isolation, they must always travel in groups of 2 to 3 quarks.

9. As described above, the four forces in relativistic quantum mechanics are mediated by the exchange of integer-spin particles (bosons).

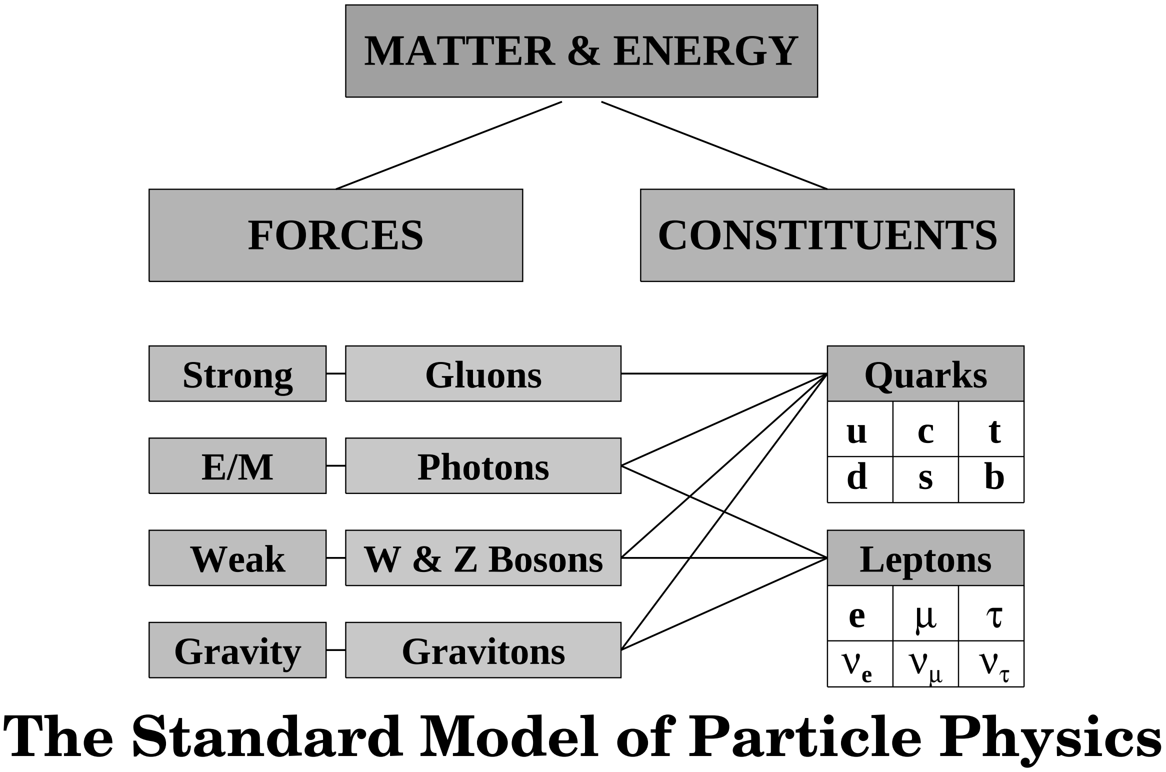

Figure VI–3: The Standard Model is the current best description of the subatomic world.

a) Of the four forces, only gravity gives rise to attractive forces between *like* particles (same type of color charge, electric charge, weak charge, or mass).

b) This difference arises because the graviton is spin 2, whereas the gluon, photon, and weakon are spin 1 (see Table VI-1).

10. This description of elementary and field particles is called the **Standard Model**. A graphical representation of this model is shown in Figure VI-3.

11. Getting back to angular momentum, *spin* can take on any half-integer or integer value for s. As mentioned above, each particle has its own specific spin (see Table VI-1) that remains *fixed.* In contrast, the *orbital* angular momentum quantum number ℓ can take on any (integer) value you please, and will change from one to another when the system is perturbed.

Table VI–1: Spin quantum numbers for a sample of elementary and field particles.

Common Name	**Symbol**†	**Particle Type**	**Spin (s)**	**Spin Family**
Pion	π^+	meson	0	boson
	π^0	meson	0	boson
Electron	e^-	lepton	$\frac{1}{2}$	fermion
Muon	μ^-	lepton	$\frac{1}{2}$	fermion
Neutrino	ν_e	lepton	$\frac{1}{2}$	fermion
Proton	p	baryon	$\frac{1}{2}$	fermion
Neutron	n	baryon	$\frac{1}{2}$	fermion
Gluon	G	field	1	boson
Photon	γ	field	1	boson
Weakon	W	field	1	boson
Delta	Δ^+	baryon	$\frac{3}{2}$	fermion
Graviton	g	field	2	boson

† – The superscript in the symbol corresponds to the charge of the particle: '+' = positive, '–' = negative, '0' = neutral. Symbols with no superscript are neutral, except for the proton which is positively charged, and the weakons which can have a +, –, or no electric charge.

E. Spin 1/2 Particles.

1. In the science of quantum mechanics, particles with $s = 1/2$ are the most important, since these are the particles that make up ordinary matter (protons, neutrons, and electrons), as well as all quarks and leptons.

2. There are just *two* eigenstates for these type of fermions:

a) $|\frac{1}{2}\,\frac{1}{2}\rangle \equiv$ the **spin up** (↑) state.

b) $|\frac{1}{2}\,(-\frac{1}{2})\rangle \equiv$ the **spin down** (↓) state.

3. Using these as basis vectors, the general state of a spin-1/2 particle can be expressed as a two-element column matrix called a

spinor:

$$\chi = \begin{pmatrix} a \\ b \end{pmatrix} = a\chi_+ + b\chi_-, \qquad \text{(VI-64)}$$

with

$$\chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \text{(VI-65)}$$

representing spin up, and

$$\chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \text{(VI-66)}$$

for spin down.

4. The spin operators become 2 × 2 matrices, which we can work out by noting their effect on χ_+ and χ_-.

a) Eq. (VI-60) says

$$S^2\chi_+ = \frac{3}{4}\hbar^2\chi_+ \,; \qquad S^2\chi_- = \frac{3}{4}\hbar^2\chi_- \,; \qquad \text{(VI-67)}$$

$$S_z\chi_+ = \frac{\hbar}{2}\chi_+ \,; \qquad S_z\chi_- = -\frac{\hbar}{2}\chi_- \,. \qquad \text{(VI-68)}$$

b) Eq. (VI-62) says

$$S_+\chi_- = \hbar\chi_+ \,; \quad S_-\chi_+ = \hbar\chi_- \,; \quad S_+\chi_+ = S_-\chi_- = 0 \,. \qquad \text{(VI-69)}$$

c) Now, $S_\pm = S_x \pm iS_y$, so

$$S_x = \frac{1}{2}(S_+ + S_-) \quad \text{and} \quad S_y = \frac{1}{2i}(S_+ - S_-) \,. \qquad \text{(VI-70)}$$

d) Using the above equations, it is easy to show that

$$S_x\chi_+ = \frac{\hbar}{2}\chi_- \,; \qquad S_x\chi_- = \frac{\hbar}{2}\chi_+ \,; \qquad \text{(VI-71)}$$

$$S_y\chi_+ = -\frac{\hbar}{2i}\chi_- \,; \qquad S_y\chi_- = \frac{\hbar}{2i}\chi_+ \,. \qquad \text{(VI-72)}$$

e) Thus in matrix form, we can write

$$S^2 = \frac{3}{4}\hbar^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} ; \quad S_+ = \hbar \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} ; \quad S_- = \hbar \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} ; \tag{VI-73}$$

while

$$S_x = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} ; \quad S_y = \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} ; \quad S_z = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{VI-74}$$

f) Finally, let's define the **Pauli spin matrices**, $\boldsymbol{\sigma}$, such that the eigenvector $\boldsymbol{S} = (\hbar/2)\boldsymbol{\sigma}$, then

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} ; \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} ; \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{VI-75}$$

g) Notice that S_x, S_y, S_z, and S^2 are all *Hermitian* (as they should be, since they represent observables). However, S_+ and S_- are *not* Hermitian (evidently they are not observable).

5. Based on Eqs. (VI-65), (VI-66), and (VI-68), the *eigenspinors* of S_z are

$$\chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix} , \ (\text{eigenvalue: } +\frac{1}{2}) ; \quad \chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix} , \ (\text{eigenvalue: } -\frac{1}{2}) . \tag{VI-76}$$

a) If you measure S_z on a particle in the general state χ (see Eq. VI-64), you get $+\hbar/2$, with probability $|a|^2$, or $-\hbar/2$, with probability $|b|^2$.

b) Since these are the only possibilities,

$$|a|^2 + |b|^2 = 1 \tag{VI-77}$$

(*i.e.*, the spinor must be normalized).

6. Bound electrons can carry out a **spin-flip transition**. When an electron in the ground state has a “spin” *parallel* to the proton’s “spin” (*i.e.*, both are spinning in the same direction), it has a slightly higher energy than when the spins are *antiparallel* (*i.e.*, the electron is spinning in the opposite direction with respect to the proton).

a) An electron in the parallel state can spontaneously decay to the antiparallel state giving rise to an emission line at 21-cm at radio wavelengths.

b) This transition is highly forbidden however (it has a very low oscillator strength) with an Einstein A-value of $A_{21} = 2.85 \times 10^{-15}$ sec^{-1} (remember the Lyman-α transition has an Einstein A-value of $A_{21} = 6.62 \times 10^{8}$ sec^{-1}).

c) Such a transition is called a **hyperfine** transition $\Longrightarrow$ transitions involving electron spin-flips.

d) Of course, hydrogen can also absorb a 21-cm photon producing an absorption line at this wavelength.

e) Much of the interstellar medium is composed of cold hydrogen gas. As such, the structure of the Milky Way Galaxy has been mapped via this 21-cm line (note that photons at this wavelength are unaffected by dust, and as such, can travel great distances through the galactic plane which is a very dusty place).

Example VI–3. Suppose a spin 1/2 particle is in the state

$$\chi = \frac{1}{\sqrt{6}} \begin{pmatrix} 1+i \\ 2 \end{pmatrix} .$$

What are the expectation values for S_x and S_y? What is the probability of getting $\hbar/2$ and the probability of getting $-\hbar/2$ if you measure S_z?

Solution:

Since we are working with vectors here, we use Eq. (IV-41) to define the inner product of two vectors:

$$\langle\chi|\chi\rangle = \boldsymbol{\chi}^\dagger\,\boldsymbol{\chi}\,,$$

where $\chi^\dagger = \tilde{\chi}^*$. As such, the expectation value for spin is determined from $\langle S_i\rangle = \chi^\dagger S_i \chi$, where one substitutes x or y for the i in this equation. Then

$$\begin{aligned}
\langle S_x\rangle &= \chi^\dagger S_x \chi = \begin{pmatrix} \frac{1-i}{\sqrt{6}} & \frac{2}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} 0 & \hbar/2 \\ \hbar/2 & 0 \end{pmatrix} \begin{pmatrix} (1+i)/\sqrt{6} \\ 2/\sqrt{6} \end{pmatrix} \\
&= \begin{pmatrix} \frac{1-i}{\sqrt{6}} & \frac{2}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} \hbar/\sqrt{6} \\ (1+i)\hbar/(2\sqrt{6}) \end{pmatrix} = \frac{\hbar}{6}(1-i) + \frac{\hbar}{6}(1+i) \\
&= \frac{\hbar}{6} - \frac{\hbar}{6}i + \frac{\hbar}{6} + \frac{\hbar}{6}i = \frac{\hbar}{3}\,,
\end{aligned}$$

and

$$\begin{aligned}
\langle S_y\rangle &= \chi^\dagger S_y \chi = \begin{pmatrix} \frac{1-i}{\sqrt{6}} & \frac{2}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} 0 & -i\hbar/2 \\ i\hbar/2 & 0 \end{pmatrix} \begin{pmatrix} (1+i)/\sqrt{6} \\ 2/\sqrt{6} \end{pmatrix} \\
&= \begin{pmatrix} \frac{1-i}{\sqrt{6}} & \frac{2}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} -i\hbar/\sqrt{6} \\ (i-1)\hbar/(2\sqrt{6}) \end{pmatrix} = \frac{\hbar}{6}(-i-1) + \frac{\hbar}{6}(i-1) \\
&= -i\frac{\hbar}{6} - \frac{\hbar}{6} + i\frac{\hbar}{6} - \frac{\hbar}{6} = -\frac{\hbar}{3}\,.
\end{aligned}$$

Following the formalism given in §VI.E.5, if you measure S_z, the probability of obtaining $\hbar/2$ is

$$|a|^2 = \left|\frac{(1+i)}{\sqrt{6}}\right|^2 = \frac{(1-i)(1+i)}{6} = \frac{1+i-i+1}{6} = \frac{2}{6} = \frac{1}{3}\,,$$

and the probability of obtaining $-\hbar/2$ is

$$|b|^2 = \left|\frac{2}{\sqrt{6}}\right|^2 = \frac{4}{6} = \frac{2}{3}\,.$$

Finally, checking that Eq. (VI-77) is satisfied, we have

$$|a|^2 + |b|^2 = \frac{1}{3} + \frac{2}{3} = 1\,. \qquad \surd$$

Example VI–4. An electron is in the spin state

$$\chi = A \begin{pmatrix} 3i \\ 4 \end{pmatrix} .$$

(a) Determine the normalization constant A.

(b) Find the expectation values of S_x, S_y, and S_z.

(c) Find the "uncertainties" σ_{S_x}, σ_{S_y}, and σ_{S_z}.

(d) Confirm that your results are consistent with all three uncertainty principles (see Eq. VI-8 and its cyclic permutations — only with S in place of L).

Solution (a):

As always, we determine the constant A by normalizing χ, so

$$1 = \chi^\dagger \chi = |A|^2 \; [(-3i)(3i) + (4)(4)] = |A|^2 \, (9+16) = 25 \, |A|^2 \,,$$

so

$$\boxed{A = \frac{1}{5} \;.}$$

Solution (b):

Using the definition of the expectation value, we get

$$\langle S_x \rangle \;\; = \;\; \chi^\dagger S_x \chi = \frac{1}{25}\frac{\hbar}{2} \begin{pmatrix} -3i & 4 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 3i \\ 4 \end{pmatrix}$$

$$
\begin{aligned}
&= \frac{\hbar}{50} \begin{pmatrix} -3i & 4 \end{pmatrix} \begin{pmatrix} 4 \\ 3i \end{pmatrix} = \frac{\hbar}{50} (-12i + 12i) \\
&= \boxed{\; 0 \;.} \\
\langle S_y \rangle &= \chi^\dagger S_y \chi = \frac{1}{25} \frac{\hbar}{2} \begin{pmatrix} -3i & 4 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 3i \\ 4 \end{pmatrix} \\
&= \frac{\hbar}{50} \begin{pmatrix} -3i & 4 \end{pmatrix} \begin{pmatrix} -4i \\ -3 \end{pmatrix} = \frac{\hbar}{50} (-12 - 12) = -\frac{24}{50} \hbar \\
&= \boxed{\; -\frac{12}{25} \hbar \;.}
\end{aligned}
$$

$$
\begin{aligned}
\langle S_z \rangle &= \chi^\dagger S_z \chi = \frac{1}{25} \frac{\hbar}{2} \begin{pmatrix} -3i & 4 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 3i \\ 4 \end{pmatrix} \\
&= \frac{\hbar}{50} \begin{pmatrix} -3i & 4 \end{pmatrix} \begin{pmatrix} 3i \\ -4 \end{pmatrix} = \frac{\hbar}{50} (9 - 16) \\
&= \boxed{\; -\frac{7}{50} \hbar \;.}
\end{aligned}
$$

Solution (c):

For spin-1/2 systems, it is always true that

$$\langle S_x^2 \rangle = \langle S_y^2 \rangle = \langle S_z^2 \rangle = \frac{\hbar^2}{4} \,,$$

so

$$\sigma_{S_x}^2 = \langle S_x^2 \rangle - \langle S_x \rangle^2 = \frac{\hbar^2}{4} - 0 \,,$$

$$\boxed{\; \sigma_{S_x} = \frac{\hbar}{2} \;.}$$

$$\sigma_{S_y}^2 = \langle S_y^2 \rangle - \langle S_y \rangle^2 = \frac{\hbar^2}{4} - \left(\frac{12}{25} \right)^2 \hbar^2 = \frac{\hbar^2}{2500} (625 - 576) = \frac{49}{2500} \hbar^2 \,,$$

$$\boxed{\; \sigma_{S_y} = \frac{7}{50} \hbar \;.}$$

$$\sigma_{S_z}^2 = \langle S_z^2 \rangle - \langle S_z \rangle^2 = \frac{\hbar^2}{4} - \left(\frac{7}{50}\right)^2 \hbar^2 = \frac{\hbar^2}{2500}\,(625 - 49) = \frac{576}{2500}\,\hbar^2\,,$$

$$\boxed{\sigma_{S_z} = \frac{12}{25}\,\hbar\ \ .}$$

Solution (d):

Here, we want to show that

$$\sigma_{S_i}\,\sigma_{S_j} \geq \frac{\hbar}{2}\,\left|\langle S_k \rangle\right|\ ,$$

where indices i, j, and k represent the cyclic permutations of the Cartesian coordinates. As such,

$$\begin{aligned}
\sigma_{S_x}\,\sigma_{S_y} &= \frac{\hbar}{2}\,\frac{7\hbar}{50} &\overset{?}{\geq}\ \frac{\hbar}{2}\,\left|\langle S_z \rangle\right| &= \frac{\hbar}{2}\,\frac{7\hbar}{50} &&\text{(right \underline{at} the uncertainty limit)},\\
\sigma_{S_y}\,\sigma_{S_z} &= \frac{7\hbar}{50}\,\frac{12\hbar}{25} &\overset{?}{\geq}\ \frac{\hbar}{2}\,\left|\langle S_x \rangle\right| &= 0 &&\text{(obeys the uncertainty limit)},\\
\sigma_{S_z}\,\sigma_{S_x} &= \frac{12\hbar}{25}\,\frac{\hbar}{2} &\overset{?}{\geq}\ \frac{\hbar}{2}\,\left|\langle S_y \rangle\right| &= \frac{\hbar}{2}\,\frac{12\hbar}{25} &&\text{(right \underline{at} the uncertainty limit)}.
\end{aligned}$$
