Collatz meets Fibonacci

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Collatz conjecture

The unresolved Collatz conjecture from 1937 states that when the function

$$f(x) = \begin{cases} x/2 & \text{if } x \equiv 0 \pmod 2 \\ 3x+1 & \text{if } x \equiv 1 \pmod 2 \end{cases}$$

is iterated on an initial positive integer x we eventually reach the cycle $(1, 4, 2)$.

- Starting from an initial value x, the sequence of iterates: $x, f(x), f(f(x)), \ldots,$ behaves at first irregularly before its eventual apparent inevitable decline from some power of 2 down to the three element cycle.
- For example, starting the iteration at 12 provides the sequence:

$$\underbrace{12,\ 6,\ 3,\ 10,\ 5}_{\text{trace}},\ 16,\ 8,\ 4,\ 2,\ 1,\ 4,\ 2,\ 1,\ \ldots.$$

- The sequence of numbers up until the first power of two is the interesting phase of the iteration, which we will call its trace.

Producing permutations

- The elements of a trace from an initial value x are all distinct, and we produce a Collatz permutation, $C(x)$, by replacing the i^{th} smallest number of the trace by i.
- So $C(12) = 53142$.
- Clearly, the map C is not one to one. For instance:

$$1 = C(5) = C(21) = C(85) = \cdots = C((2^{2k}-1)/3) = \cdots$$

- However, as the length of $C(x)$ increases, coincidences become more rare, e.g. the only other $x \le 1000$ with $C(x) = C(12)$ is $x = 908$.

Enumerating Collatz permutations

Among the permutations of length n, how many are Collatz permutations? Considering only those $x \le 10^8$ for which the length of $C(x)$ is at most 7 produces the following table.

Length	Collatz permutations
1	1
2	1
3	2
4	3
5	5
6	8
7	13

As the reader will have noticed the values in the table are the Fibonacci numbers.

Types of traces

- Two kinds of steps occur in a trace depending on the parity of the argument:
  - u: up steps $(x \mapsto 3x+1)$ when x is odd
  - d: down steps $(x \mapsto x/2)$ when x is even
- We call the resulting sequence of u's and d's the type of the trace. A type of a trace necessarily satisfies:
  - Two up steps can never occur consecutively since $3x+1$ is even when x is odd.
  - The last symbol in such a sequence must be a d, since there is a "hidden" u occurring next to take us to a power of 2. Thus the number of traces of a given length are given by the Fibonacci numbers.
- In order to show that there are at least Fibonacci-many Collatz permutations of length n it will be enough to show that any sequence of u's and d's satisfying the necessary conditions above actually occurs as the type of some trace.

Witnesses for a type

- The final element of a trace is a number of the form $(A-1)/3$ where $A = 2^a$. Define the inverses of u and d:

$$U(x) = u^{-1}(x) = (x-1)/3 \quad \text{and} \quad D(x) = d^{-1}(x) = 2x.$$

- A witness for a type, σ, is an $A = 2^a$ such that there is a trace ending at $(A-1)/3$ with type σ. For example, to find a witness for the type $uddud$ requires at least that $UDDUDU(A)$ be an integer. Note here that the "hidden" u has become explicit.
- Now

$$UDDUDU(A) = \frac{8A-29}{27}.$$

In order for this to be an integer requires $A = 7 \pmod{27}$, and remembering that $A = 2^a$, the least solution to this is $A = 2^{16} = 65536$.
- Unraveling the applications of U's and D's leads to $x = 19417$. The trace of x is

$$19417,\ 58252,\ 29126,\ 14563,\ 43690,\ 21845,$$

and $C(x) = 264153$.

At least Fibonacci

It turns out that the initial necessary integrality condition is also sufficient and that infinitely many witnesses always exist.

Proposition 1 *If a type σ contains k u's then there is a single congruence of the form $A = c \pmod{3^{k+1}}$ which must be satisfied in order that a trace of type σ ends with witness A. Consequently, there is a least witness $A = 2^a$ with $a \le 2 \cdot 3^k$, and a general witness is of the form 2^{a+jd} where j is a nonnegative integer and $d = 2 \cdot 3^k$.*

This proposition shows that every potential type has a witness and thereby proves that there are at least Fibonacci-many Collatz permutations of length n.

Excess

There are exactly Fibonacci-many Collatz permutations of length n for $n \le 14$ but for greater n there are more.

length	#perms	length	#perms	excess
1	1	15	611	1
2	1	16	989	2
3	2	17	1600	3
4	3	18	2587	3
5	5	19	4185	4
6	8	20	6771	6
7	13	21	10953	7
8	21	22	17720	9
9	34	23	28669	12
10	55	24	46383	15
11	89	25	75044	19
12	144	26	121417	24
13	233	27	196448	30
14	377	29	317850	39
		30	832101	61
		31	1346346	77
		32	2178405	96
		33	3524700	122

The first ET

- The first type that is associated with more than one Collatz permutation (an excess type or ET) is $\sigma = uddudududdddd$ which has the integrality condition

$$2^a = 16 \bmod 729.$$

- The smallest solution is $a = 4$ corresponding to the trace $9, 28, 14, 7, 22, 11, 34, 17, 52, 26, 13, 40, 20, 10, 5$ and the first permutation below.
- However, the next solution to the integrality condition is $a = 490$, giving an initial number with 440 digits producing the second permutation below.

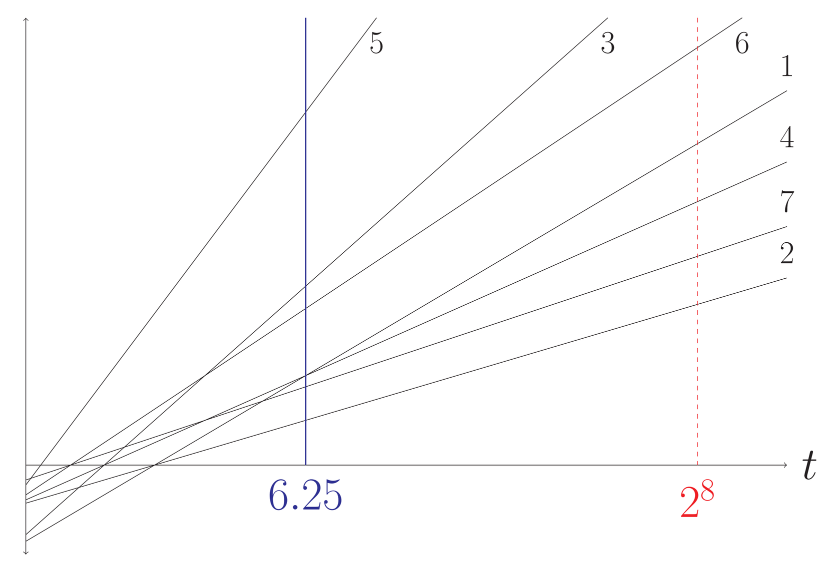
③ 12 7 2 10 5 13 8 15 11 6 14 9 ④ 1
④ 12 7 2 10 5 13 8 15 11 6 14 9 ③ 1

Explaining the ET's

- A potential witness t for the type $\sigma = dudud d$ generates the trace

$$DUDUDDU(t),\ UDUDDU(t),\ DUDDU(t), \ldots, DU(t),\ U(t).$$

We can think of these as linear functions in t. We find a witness for the type where a vertical line $t = A = 2^a$ has intersection points with these lines at integer heights.
- We can then read off the relative order of the lines at this point according to the order they are crossed as we move up the line $t = A$ from the t-axis.



- The labels on each line correspond to the point that the corresponding element would occur in a trace and so the associated permutation is 4163752.
- Consequently we see that if we have two potential witnesses such that there is an intersection point between two lines in the corresponding family lying between them, that they will determine different permutations.

At most $2\times$Fibonacci

- Recall that the type $\sigma = uddudududdddd$ had witnesses $2^4, 2^{490}, \ldots$. The greatest abscissa of an intersection point for the lines arising from this sequence is approximately 44.04. This explains why we get two different Collatz permutations.
- It turns out that we can get at most two permutations from each type:

Proposition 2 *For any type σ there are at most two distinct permutations $C(x)$ arising from x of type σ.*

Open problems and conjectures

- How exactly does the number e_n of ET's of length n behave? The data suggests that it might be something like "half Fibonacci rate" i.e. $e_n \sim e_{n-2} + e_{n-4}$.
- We always get $c = 16$ in the integrality conditions for ET's. Is this just bias in the data that we currently have?
- We have an extrinsic way of creating the Collatz permutations: run the Collatz process and see what comes out. Is there an intrinsic way to recognize these permutations?

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