Patterns and Statistics for Set Partitions

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Partitions and Sub-Partitions

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Partitions are composed of *blocks*, denoted $B_i$. We will use the notation: $B_1/B_2/\ldots/B_k \vdash [n]$. 

For example, let $[4] = \{1, 2, 3, 4\}$ and $\sigma = 1/24/3$. Then $\sigma$ is a partition of $[4]$, and $24$ is a block of $\sigma$.

If $S \subseteq [n]$ and $\sigma \vdash [n]$, then the partition $\sigma'$ of $S$ obtained by intersecting the blocks of $\sigma$ with $S$ is called a *subpartition* of $\sigma$. For example, if $\sigma = 14/236$ then $\sigma' = 26/4$ is the subpartition obtained by intersecting the blocks of $\sigma$ with $\{2, 4, 6\}$.
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For example, if $\sigma = 14/236/5$ then $\sigma' = 26/4$ is the subpartition obtained by intersecting the blocks of $\sigma$ with $\{2, 4, 6\}$. 
We define a standardization map \( st(\sigma) \) that maps the smallest integer in the subpartition to 1, the second smallest to 2, etc. We say \( \sigma \) contains \( \pi \), which we call the pattern, if there is a subpartition \( \sigma' \) such that \( st(\sigma') = \pi \). Otherwise we say \( \sigma \) avoids \( \pi \).
Pattern Containment and Avoidance

We define a standardization map $st(\sigma)$ that maps the smallest integer in the subpartition to 1, the second smallest to 2, etc. We say $\sigma$ contains $\pi$, which we call the pattern, if there is a subpartition $\sigma'$ such that $st(\sigma') = \pi$. Otherwise we say $\sigma$ avoids $\pi$.

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For example, if $\sigma = 14/236/5$ then $\sigma$ contains $\pi = 13/2$ since $st(26/4) = 13/2$. But $\sigma$ avoids $\pi = 123/4$ since there are no three numbers in a block with a larger number in another block.
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For $n \geq 0$, we denote the set of partitions which avoid $\pi$ by

$$\Pi_n(\pi) = \{\sigma \in \Pi_n : \sigma \text{ avoids } \pi\}.$$
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\[
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The cardinalities \( |\Pi_n(\pi)| \) were determined by Sagan [2] for patterns for length = 3.
Standard Form and Restricted Growth Functions

We say a partition, $\sigma = B_1/B_2/.../B_k$, is in standard form if

$$\min(B_1) < \min(B_2) < ... < \min(B_k).$$

A restricted growth function (RGF) is a sequence of positive integers, $w = a_1...a_n$, such that $a_1 = 1$ and for all $k \geq 2$,

$$a_k \leq 1 + \max(1, 2, ..., a_{k-1}).$$
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\]

For example, \( 137/24/56 \mapsto 1212331 \).
We are studying four statistics on RGFs defined by Michelle Wachs and Dennis White [3].

\[ \text{lb}(w), \text{ls}(w), \text{rb}(w), \text{rs}(w) \]

- \( \text{lb}(w) \) is defined as the sum of \( \text{lb}_i(w) \) over all \( i \), where \( \text{lb}_i(w) \) is the number of integers to the left of \( w_i \) which are also bigger than \( w_i \). (Note that multiple copies of the same integer are only counted once.)
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\( \text{lb}(w), \text{ls}(w), \text{rb}(w), \text{rs}(w) \), with “l” meaning left, “r” meaning right, “b” meaning bigger, and “s” meaning smaller.
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Define the lb statistic

\[
\text{lb}(w) = \sum_{i} \text{lb}_i(w)
\]

where \( \text{lb}_i(w) \) is the number of integers to the left of \( w_i \) which are also bigger than \( w_i \). (Note that multiple copies of the same integer are only counted once.)
For example, suppose $w = 1 2 2 1 3 1 2$. Then $\text{lb}(w) = 0 + 0 + 0 + 1 + 0 + 2 + 1 = 4$. We can then use each statistic to create a generating function (polynomial). For lb, $LB_n(\pi) = LB_n(\pi; q) = \sum_{\sigma \in \Pi_n(\pi)} q^{\text{lb}(w(\sigma))}$. Note that the coefficient of $q^k$ will be the number of partitions in the avoidance class of $\pi$ that have an lb of $k$. Other statistics have polynomials defined in a similar manner.
Statistics

For example, suppose

\[ w = 1 2 2 1 3 1 2 \]

Then

\[ \text{lb}(w) = 0 + 0 + 0 + 1 + 0 + 2 + 1 \]

So then, \( \text{lb}(w) = 4 \).
For example, suppose
\[ w = 1 \ 2 \ 2 \ 1 \ 3 \ 1 \ 2 \]
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<table>
<thead>
<tr>
<th>Statistics</th>
<th>Set Partition</th>
</tr>
</thead>
<tbody>
<tr>
<td>LB</td>
<td>1 + (\sum_{k=0}^{n-2} \binom{n-1}{k+1} q^k)</td>
</tr>
<tr>
<td>LS</td>
<td>(\sum_{k=0}^{n-1} \binom{n-1}{k} q^k)</td>
</tr>
<tr>
<td>RB</td>
<td>(\sum_{k=0}^{n-1} \binom{n-1}{k} q^k)</td>
</tr>
<tr>
<td>RS</td>
<td>1 + (\sum_{k=0}^{n-2} \binom{n-1}{k+1} q^k)</td>
</tr>
</tbody>
</table>
Characterizations For 1/23

<table>
<thead>
<tr>
<th>Statistics</th>
<th>Set Partition</th>
</tr>
</thead>
<tbody>
<tr>
<td>LB</td>
<td>$1 + \sum_{k=0}^{n-2} (n - k - 1)q^k$</td>
</tr>
<tr>
<td>LS</td>
<td>$q^{T_{n-1}} + \sum_{i=0}^{n-2} (i + 1)q^{T_i}$</td>
</tr>
<tr>
<td>RB</td>
<td>$q^{T_{n-1}} + \sum_{j=1}^{n-1} \sum_{i=0}^{n-j-1} q^{i(n-j-1)+T_{n-j-2+i}}$</td>
</tr>
<tr>
<td>RS</td>
<td>$1 + \sum_{k=0}^{n-2} (n - k - 1)q^k$</td>
</tr>
</tbody>
</table>

Note that $T_n$ is the $n$th triangular number, which is the sum $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$. 
## Characterizations For 12/3

<table>
<thead>
<tr>
<th>Statistics</th>
<th>Set Partition</th>
</tr>
</thead>
</table>
| **LB**     | \[
\left\lfloor \frac{(n-1)^2}{4} \right\rfloor \sum_{k=0} D_k q^k
\]
| **LS**     | \[q^{T_{n-1}} + \sum_{j=1}^{n-1} \sum_{i=1}^{n-j} q^{T_{n-j-1}+j(i-1)}\]
| **RB**     | \[q^{T_{n-1}} + \sum_{j=1}^{n-1} (n - j)q^{T_{n-j-1}}\]
| **RS**     | \[1 + \sum_{k=0}^{n-2} (n - 1 - k)q^k\]

\[D_k = \#\{d : d \mid k \text{ and } d + \frac{k}{d} + 1 \leq n\}\]

Note that \(D_k = \tau(k)\) for \(k \leq n - 2\).
## Characterizations For 13/2

<table>
<thead>
<tr>
<th>Statistics</th>
<th>Set Partition</th>
</tr>
</thead>
<tbody>
<tr>
<td>LB</td>
<td>$2^{n-1}$</td>
</tr>
<tr>
<td>LS[1]</td>
<td>$\prod_{i=1}^{n-1} (1 + q^i)$</td>
</tr>
<tr>
<td>RB[1]</td>
<td>$\prod_{i=1}^{n-1} (1 + q^i)$</td>
</tr>
<tr>
<td>RS</td>
<td>$2^{n-1}$</td>
</tr>
</tbody>
</table>
Characterizations For 123

<table>
<thead>
<tr>
<th>Statistics</th>
<th>Set Partition</th>
</tr>
</thead>
<tbody>
<tr>
<td>LB</td>
<td>Partial Characterization, See Below</td>
</tr>
<tr>
<td>LS</td>
<td>Partial Characterization</td>
</tr>
<tr>
<td>RB</td>
<td>Partial Characterization</td>
</tr>
<tr>
<td>RS</td>
<td>Partial Characterization</td>
</tr>
</tbody>
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The polynomial $LB_n(123)$ has degree $\left\lfloor \frac{n(n-1)}{6} \right\rfloor$. 
Let $P$ be a set of partitions. We let

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$$\Pi_n(P) = \{\sigma : \sigma \text{ partitions } \llbracket n \rrbracket \text{ and avoids every } \pi \in P\}$$

We have characterized all of the statistics for patterns of length 3 pair wise, as well as some of length four.
Theorem (D,D,G,G,P,R,S)\n
The polynomial $LB_n(123)$ has degree equal to $\left\lfloor n \left( n - 1 \right) / 6 \right\rfloor$.

First, note that $\sigma = B_1 / ... / B_k$ avoids $123$ if and only if $|B_i| \leq 2$ for all $i$. So the corresponding RGF, $w$, has every entry repeated at most twice.

Define the initial run of $w = a_1 a_2 ... a_n$ to be the longest initial string of the form $12...i$. 

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We first show that a word of the form \( w = 123 \ldots ia_{i+1} \ldots a_n \) with \( a_{i+1} \ldots a_n \leq i \) and \( \{a_{i+1}, \ldots, a_n\} = [n - i] \) maximizes the lb statistic.
Proving the Partial Characterization of $LB_n(123)$

We first show that a word of the form $w = 123 \ldots ia_{i+1} \ldots a_n$ with $a_{i+1} \ldots a_n \leq i$ and $\{a_{i+1}, \ldots, a_n\} = [n – i]$ maximizes the lb statistic.

Proof.

To do this, we show that the numbers after the initial run $123 \ldots i$ must be less than or equal to $i$, and then show that they form a permutation of the numbers $[1, n – i]$.

To prove the former, assume towards a contradiction that there exists an element $a_j$ with $a_j \geq i + 1$. By the structure of the RGF, there must be at least one element equal to $i + 1$. Locate the first such element. By swapping $a_{i+1}$ and the first $i + 1$, the lb actually increases. Because of this, we know that the original RGF was not maximal, which is a contradiction.

Proving the second claim is a similar process.
We first show that a word of the form \( w = 123 \ldots ia_{i+1} \ldots a_n \) with \( a_{i+1} \ldots a_n \leq i \) and \( \{a_{i+1}, \ldots, a_n\} = [n - i] \) maximizes the \( \text{lb} \) statistic.

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Now that we know that our word can be put into the form $w = 123 \ldots ia_{i+1} \ldots a_n$ with $a_{i+1} \ldots a_n \leq i$ and \{a_{i+1} \ldots a_n\} = [n - i], we must pick the value of $i$ to maximize lb.

Proof.

Each element $j$ after our initial run contributes $i - j$ to lb. So $lb(w) = \sum_{j=1}^{n-i} (i - j)$. Since this is an arithmetic progression, this simplifies to:

$$f(i) = (i-1) + (2i - n) = \frac{-3i^2 + 4ni + i - n^2}{2}.$$ 

To maximize, we compute the derivative of $f$ with respect to $i$ to get $\left(-\frac{6i + 4n + 1}{2}\right)$ and set this equal to zero to obtain $i = \frac{4n + 1}{6}$. 

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Patterns and Statistics for Set Partitions
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To maximize, we compute the derivative of \( f \) with respect to \( i \) to get
\[
(-6i + 4n + 1)/2 \]
and set this equal to zero to obtain
\[
i = (4n + 1)/6.
\]
Since we only want integer values of $i$, we utilize the ceiling and floor functions to find the nearest whole number on a case to case basis as follows:

\[
\begin{align*}
    i &= \lfloor \frac{4n+1}{6} \rfloor \\
    i &= \lceil \frac{4n+1}{6} \rceil \\
\end{align*}
\]

for integer values of $k$. Finally, by substituting our values of $i$ into the equation for the $\text{LB}_n(123)$ sum and using algebraic manipulations, we come up with the maximum value of $\text{LB}_n$ being $\lfloor \frac{n(n-1)}{6} \rfloor$.
Since we only want integer values of $i$, we utilize the ceiling and floor functions to find the nearest whole number on a case to case basis as follows:

\[
\begin{align*}
    i &= \left\lfloor \frac{4n+1}{6} \right\rfloor & \text{if } n = 3k, \\
    i &= \left\lceil \frac{4n+1}{6} \right\rceil & \text{if } n = 3k + 1, \\
    i &= \left\lfloor \frac{4n+1}{6} \right\rfloor \text{ or } \left\lceil \frac{4n+1}{6} \right\rceil & \text{if } n = 3k + 2.
\end{align*}
\]  

(1)

for integer values of $k$. 

Finally, by substituting our values of $i$ into the equation for the LB sum and using algebraic manipulations, we come up with the maximum value of $\text{LB}_n(123)$ being $\left\lfloor \frac{n(n-1)}{6} \right\rfloor$.
Proving the Partial Characterization of LB\(_n(123)\)

Since we only want integer values of \(i\), we utilize the ceiling and floor functions to find the nearest whole number on a case to case basis as follows:

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\end{align*}
\]

(1)

for integer values of \(k\). Finally, by substituting our values of \(i\) into the equation for the lb sum and using algebraic manipulations, we come up with the maximum value of lb being \(\left\lfloor n(n - 1)/6 \right\rfloor\). \(\square\)
Bijections

In looking at these avoidance classes, it can be interesting to find bijections between avoidance classes, $\Pi_n(\alpha)$ and $\Pi_n(\beta)$. More specifically, if $\pi \in \Pi_n(\alpha)$ and $\sigma \in \Pi_n(\beta)$, and $s$ and $t$ are two statistics on set partitions, we want a bijection $\phi: \Pi_n(\alpha) \mapsto \Pi_n(\beta)$ with:

$$\phi(\pi) = \sigma \Rightarrow s(\pi) = t(\sigma).$$

Finding such a bijection between $\Pi_n(\alpha)$ and $\Pi_n(\beta)$ gives

$$\sum_{\pi \in \Pi_n(\alpha)} q^{s(\pi)} = \sum_{\sigma \in \Pi_n(\beta)} q^{t(\sigma)}.$$
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In looking at these avoidance classes, it can be interesting to find bijections between avoidance classes, $\Pi_n(\alpha)$ and $\Pi_n(\beta)$. More specifically, if $\pi \in \Pi_n(\alpha)$ and $\sigma \in \Pi_n(\beta)$, and $s$ and $t$ are two statistics on set partitions, we want a bijection

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Finding such a bijection between $\Pi_n(\alpha)$ and $\Pi_n(\beta)$ gives

$$\sum_{\pi \in \Pi_n(\alpha)} q^{s(\pi)} = \sum_{\sigma \in \Pi_n(\beta)} q^{t(\sigma)}. \quad (2)$$
Bijections

Theorem (D,D,G,G,P,R,S)

There exists a bijection $\phi : \Pi_n(1/2/3) \mapsto \Pi_n(1/2/3)$ that satisfies,

$\forall \pi \in \Pi_n(1/2/3), \, \text{lb}(\pi) = \text{rs}(\phi(\pi))$. 

Proof.
For a given partition $\pi \in \Pi_n(1/2/3)$, let $w = a_1 a_2 \ldots a_n$ be its associated RGF, which consists of only ones and twos.

Letting $\phi(w) = w' = a_1 (3 - a_n)(3 - a_n - 1) \ldots (3 - a_2)$ gives our desired bijection.

Showing that $\phi$ is well defined is a simple process.

To show that $\phi$ takes lb to rs, examine some element $a_i$ in $w$. The reversal and complementation guarantees that a left bigger from $a_i$ in $w$ leads to a right smaller caused by the image of $a_i$ in $w'$.

Since the statistics are sums of parts, and there is a correspondence between the statistics on the parts, $\text{lb}(w) = \text{rs}(\phi(w))$. 

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For a given partition $\pi \in \Pi_n(1/2/3)$, let $w = a_1 a_2 \ldots a_n$ be its associated RGF, which consists of only ones and twos. Letting $\phi(w) = w' = a_1(3 - a_n)(3 - a_{n-1}) \ldots (3 - a_2)$ gives our desired bijection.
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Bijections

**Theorem (D,D,G,G,P,R,S)**

There exists a bijection \( \phi : \Pi_n(1/2/3) \mapsto \Pi_n(1/2/3) \) that satisfies,
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\forall \pi \in \Pi_n(1/2/3), \quad lb(\pi) = rs(\phi(\pi)).
\]

**Proof.**

For a given partition \( \pi \in \Pi_n(1/2/3) \), let \( w = a_1 a_2 \ldots a_n \) be its associated RGF, which consists of only ones and twos. Letting \( \phi(w) = w' = a_1 (3 - a_n)(3 - a_{n-1}) \ldots (3 - a_2) \) gives our desired bijection. Showing that \( \phi \) is well defined is a simple process. To show that \( \phi \) takes \( lb \) to \( rs \), examine some element \( a_i \) in \( w \). The reversal and complementation guarantees that a left bigger from \( a_i \) in \( w \) leads to a right smaller caused by the image of \( a_i \) in \( w' \).
Theorem (D,D,G,G,P,R,S)

There exists a bijection \( \phi : \Pi_n(1/2/3) \mapsto \Pi_n(1/2/3) \) that satisfies, \( \forall \pi \in \Pi_n(1/2/3), \text{lb}(\pi) = \text{rs}(\phi(\pi)) \).

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For a given partition \( \pi \in \Pi_n(1/2/3) \), let \( w = a_1 a_2 \ldots a_n \) be its associated RGF, which consists of only ones and twos. Letting \( \phi(w) = w' = a_1(3 - a_n)(3 - a_{n-1})\ldots(3 - a_2) \) gives our desired bijection. Showing that \( \phi \) is well defined is a simple process. To show that \( \phi \) takes lb to rs, examine some element \( a_i \) in \( w \). The reversal and complementation guarantees that a left bigger from \( a_i \) in \( w \) leads to a right smaller caused by the image of \( a_i \) in \( w' \). Since the statistics are sums of parts, and there is a correspondence between the statistics on the parts, \( \text{lb}(w) = \text{rs}(\phi(w)) \).
For Example

Let $\pi = 14/235 \in \Pi_n(1/2/3)$, with associated RGF

$$w = 12212.$$
A \( q \)-analogue of the Catalan numbers

The partitions avoiding 13/24 are the noncrossing partitions. Therefore \(|\Pi_n(13/24)| = C_n\). They are related to 2-colored Motzkin paths which are a Motzkin paths where each level step has been colored one of two colors.

Theorem (D,D,G,G,P,R,S)

Let \( RS_n = RS_n(13/24)\). Then this polynomial satisfies the initial condition \( RS_0 = 1 \) and the recurrence

\[
RS_n = 2 RS_{n-1} + \sum_{k=1}^{n-2} q^k RS_k RS_{n-k-1}.
\]

This generating function is the same as the one for 2-colored Motzkin paths with \( n \) vertices by area.
The bijection used to prove this recurrence for $RS_n(13/24)$ is very similar to the direct sum of permutations and exactly the same as the bijection used by Chen, Dai, Dokos, Dwyer, and Sagan to prove a conjecture of Duncan and Steingrímsson on ascent sequences.

In future work, we hope to prove the following conjecture:

**Conjecture**

We have the equality

$$RS_n(13/24) = LB_n(13/24).$$
In Conclusion

Where to next?

- Patterns of Longer Length
- Multivariate Generating Functions
- 123
- 13/24
- Polynomials Generated by Other Statistics
Thank You! Any questions?
Adam M. Goyt, Bruce E. Sagan.  
Set partition statistics and q-Fibonacci numbers  

Bruce E. Sagan.  
Pattern avoidance in set partitions  

Michelle Wachs and Dennis White.  
$p, q$-Stirling numbers and set partition statistics.  